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Research Article

THE STUDY OF OPERATIONS AND COMPOSITIONS IN FUZZY RELATIONS

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Abstract

One of the most fundamental notions in pure and applied mathematics is the concept of relations. Fuzzy relations generalize the concept of the relations in the same manner as fuzzy set generalize the fundamental idea of sets. This work presents an overview of operations, compositions and their types.

Keywords: Fuzzy relation, min-max composition, Fuzzy ordering relations etc.

INTRODUCTION

In 1965, L. A. Zadeh introduced the concept of Fuzzy Theory. Fuzzy set theory is an extension of classical set theory. A relation is a mathematical description of a situation where certain elements of sets are related to one another in some way. Relation plays very important role to obtain a structure. So, in this paper, their operations, their compositions, their types are studied.

Definition:

Fuzzy Relation: An n-ary Fuzzy relation R is a fuzzy set on $U_1 \times U_2 \times U_3 \times \ldots\ldots \times U_n$ where $U_1, U_2, \ldots, U_n$ are domains. A 2- ary fuzzy relation is also called a binary fuzzy relation. Similarly, a 3-ary is also called ternary fuzzy relation. Thus, a binary fuzzy relation (BFR) is a of the form

$R=\Sigma \frac{R(u,v)}{(u,v)}$ where (u, v) varies over u × v.

We say that R is from U to V and is indicated by R: U→ V.

Example: Let U= {a, b, c} and V= {x, y}
Then a BFR on U × V is given by

R	*X*	*Y*
a	*0.6*	*1.0*
b	*0.3*	*0.5*
c	*0.4*	*0.2*

This is called the tabular or matrix representation of R and it is very useful when dealing with FFRs. We also write R as a matrix, when the elements of the domain are understood and need not be specified explicitly.

Operations on Fuzzy Relations

Let $U_1, U_2, \ldots, U_n$ be domains and let $U = U_1 \times U_2 \times \ldots \times U_n$. Then U is also domain, by definition of Cartesian product and u ∈ U looks like $u = (u_1, \ldots, u_n)$, an n- tuple.

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We thus have PF (U) = PF ($U_1 \times U_2 \times \ldots \times Un$). This equality shows that every n-ary fuzzy relation (FR) on $U_1 \times U_2 \times U_3 \times \ldots \times U_n$ is a fuzzy set on U and vice-versa. Hence, all the usual operations applicable to fuzzy sets are applicable to FRs also. We quickly summarize them. Let $U = U_1 \times U_2 \times \ldots \times U_n$.

Equality: For R, S in PF(U). We say R = Sif and only if R(u) = S(u) for all u in U.

Containment: For R, S in PF (U). We say R⊆ Sif and only if R(u) ≤ S(u) for all u in U.

Union: For R, S in PF(U), the union of R and S, denoted by R ∪ S, is defined by$(R \cup S)(u) = max$ [R (u), S (u)] for all u in U.

Intersection

For R, S in PF (U), the intersection of R and S, denoted by R ∩ S, is defined by (R∩ S) (u) = min[R(u), S(u)] for every u in U.

Complementation: For R in PF(U), R' is defined by R' (u) = 1 – R(u) for every u in U.

α-cuts of the Fuzzy Relation

Among many if the properties of Fuzzy set carry over to fuzzy relation one of them is the concept of α-cuts and it's associated properties.

Let R be a FR on Ux V and α be such that 0<α≤ 1.Then, the α-cut of R, denoted by R_α, is defined by

$R_\alpha = \{(u, v) \mid R(u, v) \geq \alpha\}$

Note that R_α is a crisp set on U x V and hence is a crisp (binary) relation on U x V. The α-cut of R satisfy the following property called the decomposition theorem or resolution form of R.

Let R be a fuzzy relation on U x V. Then R= $\sum (\alpha R_\alpha)$ where $\sum$ is taken over all α.

The following example illustrates the above point.

Example:

Let R be a fuzzy relation on U×V given by the matrix

R= $\begin{bmatrix}0.7 & 0.4\\0.4 & 0.0\end{bmatrix}$Then $R_{0.4}=\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$ and$R_{0.7}=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

Now, (0.4× $R_{0.4}$) ∪

(0.7× $R_{0.7}$)=$\begin{bmatrix}0.4 & 0.4\\0.4 & 0\end{bmatrix}\cup\begin{bmatrix}0.7 & 0\\0 & 0\end{bmatrix}=\begin{bmatrix}0.7 & 0.4\\0.4 & 0\end{bmatrix}$ =R

This verifies the above theorem.

Composition of fuzzy Relations

Composition of fuzzy relations plays a crucial role. Composition of two fuzzy relations can be define in several different ways. In the present only two of them, namely, max-min composition and max product composition and discuss only one of them.

Max-min composition of two fuzzy relations: Let R be a BFR on U × V and S a BFR on V × W. Then, the max-min composition of R and S (that is composition of R followed by S) is a BFR on U × W, denoted by S∘R and is given by

(S∘R) (U,W)= $\max[\min\{R\ (U,V), S\ (V,W)\}]$ *w*here the maximum is taken over all v in V.

More generally, let R be in pF (U x V) and S be in pf (V x W), Where $U = U_1 \times U_2 \times \ldots.\times U_k$;$V = V_1 \times V_2 \times \ldots.\times V_m$; $W = W_1 \times W_2 \times \ldots.\times W_n$

Then, the max-min composition of R and S, denoted by S ∘ R is a fuzzy relation on U × W and is given by

(S∘R) (u, w)= $\max[\min\{R\ (u,v), S\ (v,w)\}]$

Where

$u = (u_1 \times u_2 \times u_3 \times \ldots \times u_k)$
$v = (v_1 \times v_2 \times v_3 \times \ldots \times v_m)$
$w = (w_1 \times w_2 \times \ldots.\times w_n)$

The maximum is taken over all v in V.

Note that R is a (k+m)-ary fuzzy relation S is an (m+n)-ary fuzzy relation and V is the common domain (or, called the linking domain) of R and S. This is called the compatibility condition for composition.

And finally, SoR is a (k +n)-ary fuzzy relation.

Examples

Consider the fuzzy relations R on U×V and S onV×W,

Where

U= $\{a,b,c\}$
V= $\{x,y,z\}$
And W= {&,∗},
given in matrix form by

R=$\begin{bmatrix}1.0 & 0.4 & 0.5\\0.3 & 0.0 & 0.7\\0.6 & 0.8 & 0.2\end{bmatrix}$

and
S =$\begin{bmatrix}0.7 & 0.1\\0.2 & 0.9\\0.8 & 0.4\end{bmatrix}$

Then SoR can be defined and it is fuzzy relation on U×W. Now,

(S∘ R) (a, &)= max$[min\{R\ (a,v), S\ (v,\&)\}]$ for ever v in V
= $[min\{R\ (a,x), S\ (x,\&)\}$,
$min\{R\ (a,y), S\ (y,\&)\}$,
$min\{R\ (a,z), S\ (z,\&)\}]$
= max $[\min(1,0.7)\,, \min(0.4, 0.2), \min(0.5,0.8)]$
=max$[0.7\ , 0.2\ , 0.5]$
= 0.7

Similar computation for (a, *), (b, &), (b, *), (c, &) and (c, *) yield the following matrix of SoR:

S∘R=$\begin{bmatrix}0.7 & 0.4\\0.7 & 0.4\\0.6 & 0.8\end{bmatrix}$

We now discuss a special case of composition which plays a fundamental role in fuzzy logic.

This special case deals with the composition of a fuzzy set and a fuzzy relation, as explained in the following definition.

Let A be a fuzzy set on U and R be a fuzzy relation on U×V, where,

$V = V_1 \times V_2 \times \ldots.\times V_n$Then ,

Then, the compositon of A followed by R also called the image of A under R, denoted by

RoA, and defined as:
(R∘A) (V) =max$\left[min[a(u), R(,V)]\right]$ where 'max' is taken over all u in U.

Clearly, RoA is an n-ary fuzzy relation on V (in case n =1, it is a fuzzy set on V). We can, in a similar way, define the max-product composition of A and R.

Properties of Min –Max composition

Associativity: The max-min composition is associative, i.e. $(R_1 o\ R_2)$ o R_3 =R_1 o $(R_2$ o $R_3)$

Reflectivity:

Let R be a fuzzy relation on X× X.

Then

i. R is said to be a reflexive if $\mu_R\ (x,x) = 1$,for all x∊ X
ii. R is said to be ϵ reflexive if $\mu_R\ (x,x) \geq \epsilon\ 1$,for all x∊ X

iii. R is said to be weakly reflexive if $\mu_R(x, y) \leq \mu_R(x, x)$, $\mu_R(y, x) \leq \mu_R(x, x)$ for all x , y ϵ X

Symmetry:

- A fuzzy relation R is called symmetric if $R(x, y) = R(y, x)$
- A relation R is called anti symmetric if for x≠y,

  Either $\mu_R(x, y) \neq \mu_R(y, x)$

  Or $\mu_R(x, y) = \mu_R(y, x) = 0$, for all x, y ϵ X

- A relation is called perfectly anti symmetric if for x≠y, wherever $\mu_R(x, y) > 0$, then $\mu_R(y, x) = 0$, for all x,y ϵ X.
- If R_1 is reflexive and R_2 is an arbitrary fuzzy relation from R

  $R_2 \subset R_1 \circ R_2$
- If R is reflexive, then $R \subset R_1 \circ R_2$
- If R_1 and R_2 are symmetric, then R_1 o R_2 is symmetric if R_1 o R_2 = R_2 o R_1.
- If R is symmetric, then each power of R is symmetric.
- Transitive: A fuzzy relation R is called max-min transitive if $R \circ R \subseteq R$.
- If R is symmetric and transitive, then $\mu_R(x, y) \subseteq \mu_R(x, x)$, for all x, y ∈ X.
- If R is reflexive and transitive, then $R \circ R = R$.
- If R_1 and R_2 are transitive and $R_1 \circ R_2 = R_2 \circ R_1$, then $R_1 \circ R_2$ is transitive.

Definitions

Definition: A fuzzy relation which is max-min transitive and reflexive is called a fuzzy pre-order relation.

Definition: A fuzzy relation, i.e., max – min transitive, reflexive and antisymmetric is called a fuzzy order relation.

If a relation is perfectly antisymmetric, then relation is called perfect fuzzy relation. It is also called fuzzy partial order relation.

Definition: A total fuzzy order relation or a fuzzy linear ordering is a fuzzy order relation such that for all x, y ∈ X, x ≠ y, either

$\mu_R(x, y) > 0$
or $\mu_R(y, x) > 0$.

Conclusion

Operations with fuzzy relations are important in process Fuzzy models constructed via Fuzzy relations. Relations are associations and remain at the very basis of most methodological approaches of Science and engineering.

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