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Research Article

THE CONFORMAL GEOMETRY OF SUB MANIFOLDS SPACES AND THEIR SOME APPLICATIONS TO KINEMATIC QUANTITIES OF SPACE TIMES

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Abstract

The aims of this study is to identify the conformal geometry of submanifold in a Riemannian spaces and a related some problems We followed the historical , analysis mathematical method and we found that : any Riemannian manifold is a Riemannian metric also every Riemannian 2-manifolds are conformally flat, Since for any semi Riemannian manifold, there is a natural existence of a light like subspace (hypersurface or submanifold),whose metric is degenerate, one fails to use the theory of harmonic maps of non-degenerate manifolds for the light like case and there are many physical applications of manifolds.

Keywords: Conformal Geometry, Submanifold Spaces.

1. INTRODUCTION

In this study the A manifold is a topological space where some neighborhood of a point looks like an open set in Euclidean space. We study smooth manifolds is detail as tangent bundles which are naturally related to them. We cover the element of theory of critical points of smooth functions on manifolds. We now use the preceding result to study certain substructures of manifold and topics again after studding the differential of a function. The metric on a manifold M that we considered so far comes from Urysohn's metrication theorem and also from the fact that M is embeddable in some Euclidean's space. The second Metric has nothing to do with smooth structure it may be obtained for any nice topological space.This metric comes from a Riemannian structure on M and is defined intrinsically.

2. A TOPOLOGICAL SPACE CONSISTS OF TWO OBJECTS

Definition (2.1): Anon-empty set on x and a topology Ton x. The sets in the class T are called the open sets of topological space (x, T) and the elements x are called its points. It is customary to denote the topological space(x, T) by the symbol x which is used for its underlying set of points.

Examples (2.2):

i. Let X be any metric space and let the topology be the class of all subsets of x which are open. This is called the usual topology a metric space and we say that these sets are the open sets generated by metric on the space. Metric spaces are the most important topological space and whenever we speak of a metric space as a topological space.
ii. Let X be any nonempty set and let the topology be the class of all subsets of X this is called the discrete topology on X, and any topological space whose topology is the discrete topology is called a discrete space.
iii. Let X be any infinite set and let the topology consist of empty set $\emptyset$ togerther with all subsets of X whose complements are finite. (ARTHUR *et al.,* 1973).

Theorem (2.3):

Let X be a topological space and A anarbitary subset of X then:
$\bar{A} = \{x: each\ neighbrohood\ of\ x\ intresects\ A\}$
(Georgef. Simmous 1963)

Definition (2.4) : For every subset A of A topological space E the set A with the topology whose open sets are the traces on A of the open sets in E called the subspaces A of E.

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3. CLOSED SET AND NEIGHBORHOODS IN A SUBSPACE

The formula $A - A \cap W = A \cap C_w$Where w is an open set in E, shows that closed sets in the subspace A are simply the traces on A of closed sets in E.Similarly in a subspace A the neighborhoods of a point a of A are the sets $A \cap v$ where v is a neighborhood of A in E. In fact, if A is open in E, the trace on A of every open set in E is open in E. Conversely if the trace on A of every open set in E is open in E, this is true of the trace of E on A that is of A it self. (Gustave Choquet, 1966).

Definition (3.1): A topological space S is connected if and only sets which are both open and closed are $\emptyset$ and S. (Amiyamuk, 2005).

Theorem (3.2):

A topological space S is connected if the only if it is not the union of two disjoint non-empty open set.

Proof:

Assume S is connected. Suppose $S = v_1 \cup v_2$ for open sets v_1 and $v_2$with $v_1 \cap v_2 = \emptyset$.
Then $v_1 = \overline{v_2}$, so v_1 is closed as open.
Since S is connected, either $v_1 = \emptyset$ or $v_1 = S$.
If $v_1 = S$, then $v_1 = \emptyset$, so in both cases either v_1 or v_2 must be empty.
Conversely, suppose S is not the union of two disjoint non-empty sets.

Let $v \subset S$ be both open and closed. Then $\overline{v}$ is also both open and closed and s is the union of the disjoint open sets v and $\overline{v}$. Thus either $v = \emptyset$ or $\overline{v} = \emptyset$ that is either $v = \emptyset$ or $v = s$, so S is connected. (ARTHUR A. Sagle, Ralp he, walde, 1973) .

Remark (3.3):

A subset of a topological space is said to be connected if it is connected in the relative topology.(Amiyamuk,2005) .

Definition (3.4): Let S be a set. A collection $\mathcal{v} \subset 2^S$ is a covering of S if $U_{\mathcal{v} \in \mathcal{v}} V = S$. If S is a topological space and each $\overline{v} \in v$is an open set v is called on open covering of S.A topological space S is compact if every open covering has a finite sub covering that is if for every open a covering v there exist a finite number of sets say $v_1, \dots, v_k \in \mathcal{v}$ for some k, such that $S = \bigcup_{j=1}^{k} v_j$.

Theorem (3.4):

Every closed subsets of a compact space is compact in its relative topology. (I.M singer – J.A. Thorpe 1967).

Theorem (3.5):

Every finite product of compact spaces is compact .

Proof:

By the associatively of the product topology it suffices to prove the theorem for the product of two spaces.

Let $E = X \times Y$ be the product of compact spaces $x$and $y$since x and $y$are separated, E is separated. Now Let $(w_i)_{i \in J}$ such that $m \in w_{im}$. There fore there exist open neighborhoods$v_m$and w_m of x and y in X and Y such that $v_m \times w_m \subset w_{im}$, we set $U_m = v_m \times w_m$.

But for every $x_0 \in X$, the subset $y_0 = x_0 \times Y$ of $X \times Y$ is homimorphic to Y hence compact. The $U_m, m \in y_0,$ constitute an open covering if Y_0, we can find a finite sub covering

$(\mathrm{U}_{mj})\ j \in J$where$mj = (x_0, y_j)$

We set:

$$v_{x_0} = \bigcap_{j \in J} v_{mj}$$

This is an open neighborhood of x_0 and it is clear that:

$$\bigcup_{j \in J} wi_{mj} \supset v_{x_0} \times y$$

The v_{x_0} form an open covering of x, we can find a finite subcoeving with each of the corresponding points x_0 there is associated a subfamily(wi_{mj}) of openset wi the union of these formilies is afinite family which convers E. (Amiyamuk, 2005).

Corollary (3.6):

The compact subspace of R^n is the closed and bounded subset of R^n.If A is a compact set in R^n, it is closed in R^n, on the other hand the projection of A on to each Factor $R$is compact and therefore contained in abounded interval. Hence A is contained in an interval with bounded, it is a closed subset of a finite products of a finite product of compact intervals$[a_i, b_j]$, such a product is a compact is hence so is A.

Example (3.7):

The sphere $S_{n-1}$of R^n is closed and bounded there for compact. It follows that the torus $(S_1)_p$ is also compact**.** (Gustave Choquet 1966).

4. MANIFOLDS

There are two ways to look at a differentiable manifold. Firstly it is a topological space with a structure which helps us to define differentiable functions on it, just as a topological structure on a set is designed to define continuous function on that set. Secondly it is a topological space which can be obtained by gluing together open subsets of some Euclielean space in a nice way think for example of the surface of a ball or a torus covered with small paper disks pasted together on over laps without making any crease or fold. Both the approaches to a differentiable manifold are based on the standard differentiable structure on a Euclidean space R^n.

Let us therefore recall from calculus the notion of differentiable functions on R^n.

Let $u_1, \dots, u_n$ denote the coordinate functions where $u_i = R^n \to R$ is the function mapping a point $P = (P_1, \dots, P_n)$ onto i-th coordinate P_i.

Definition (4.1): A sheaf of continuous functions on X is a map $F_x$on u which assigns to each open subset u ∈ Ua sub algebra $F_x(u)$ of the algebra C°(u) such that:

1) $F_x(\emptyset) = 0$
2) $If\ u\ ,V \in U$with$V \subseteq U$ and $F \in F_x(u)$ then $F/v \in F_x(v)$.
3) If $u \in U$ and $F: U \to R$ is a function such that each $P \in U$ has open neighborhood $v \subset U$ on which F coincides with $ag \in F_x(v)$then $F \in F_x(U)$.

We call the pair (x, f_x)an a- space. An example is provided by $X = R^n$ with the sheaf F_x as the sheaf of smooth function,
$c^\infty: U \to c^\infty(U)$.

Example (4.2):

Let (x, f_x) be a-space, Y a topological space and $\emptyset: X \to Y$ a homeomorphism .Then the sets
$F_y(U) = \{F: u \to R/Fo\,\emptyset \in F_x(\emptyset^{-1}(U))\}$.

U open in y, define a sheaf of function on Y, and $\emptyset$ is an isomorphism between the a spaces $(x, F_x) \to (Y, F_y)$

Definition (4.3): A smooth manifold of dimension n is a second countable Hausdorff a space (M, f_m) which is locally isomorphic to the a space (R, c^∞). (Amiyamuk, 2005).

Definition (4.4): A topological manifold M for which the transition maps $\emptyset_{ij} = \emptyset_j o \emptyset_i^{-1}$ for all pairs $\emptyset_i, \emptyset_j$ in the atlas are diffemorphisms is called a differentiable of smooth manifold.The transition maps are mappings between open subsets of R^n.

Definition (4.5): Let M be a topological manifold and Let D be a differentiable structure on M with maximal atlas A. then the pair (M, A_D) is called a C^∞differentiable manifold. (Abe, k, 1913).

Theorem (4.6):

Let M be a topological manifold with a $C^\infty - Atls$ A then there exist aunique differentiable structure D containing A such that $A \subset A_D$. (R-C.A.M. Vander V anderst 2007).

5. SMOOTH MAPS BETWEEN MANIFOLDS

Definition (5.1): If (M, f_m) and (N, f_N) are smooth manifolds then amorphism $f: (M, F_m) \to (N, F_N)$ is a smooth map. Explicitly a continuous map $f: M \to N$ is a smooth map if $g \in F_N(U)$ implies $gof \in F_m(F^{-1}(U)$ for every open set $U$of N. (ARTHUR *et al.,* 1973).

Lemma (5.2):

Suppose $(U_\alpha, \varphi_\alpha)$is a smooth atlas for$M. If\ F: M \to R^k$ is Function such that $Fo\varphi_\alpha^{-1}$is smooth for each α then F is smooth.

Proof:

We just need to check that $Fo\varphi^{-1}$ is smooth for any smooth chart (U, φ) in M.It suffices to show it smooth in a neighborhood of each point $X = \varphi(P)$ for any $p \in U$.There is chart $(U_\alpha, \varphi_\alpha)$in the atlas whose domain contains p. Since (U, φ) is smoothly compatible with $(U_\alpha, \varphi_\alpha)$the transition map $\varphi_\alpha o \varphi^{-1}$ is smooth on its domain of definition which includes X. Given a function $F: M \to R^k$ and a chart (U, φ) for M, the function $\hat{F}: \varphi(U) \to R^k$ defined by $\hat{F}(x) = Fo\varphi^{-1}(x)$ is called the coordinate representation of F.(P.M. cohn and G.E.H. Reuter 1970).

Example (5.3):

Define $P: S^n \to P^n$ as the restriction of

$$\prod : R^{n+1} \backslash \{0\} \to P^n \quad to \ \ S^n \ c \ R^{n+1} \backslash \{0\}$$

It is smooth map, because it is the composition $P = \pi oi$ of the maps in preceding two examples. (Joel w.Robbin 1976).

Definition (5.4): Let M be a c^∞ $-$mainfold and let $D(m)$ be the vector space of c^∞ $-$vector fields on M.

An affine connection on M is an R $-$billear map$\nabla: D(m) \times D(m) \to D(m): (X, Y) \to \nabla_x y$ satisfying $\nabla_{F_x + g_x}(z) = F\nabla_x z + g\nabla_y z$
$\nabla_x(Fy) = F\nabla_x Y + (xF)Y$

Where$F, g \in c^\infty(m)$. The operator ∇_x is called covariant differentiation relative to x. (Gustave Choquet 1966).

Definition (5.5) : Let M be a c^∞ $-$mainfold with affine connection ∇ let $\sigma = I \to M$be ac^∞ $-$curve in M with tangent vector field x, that is $x(t) = \sigma(t)$ for all t in the open interval I and let J be a closed subinterval of I.A c^∞ $-$vector field Y on σ is parallel along σ(restricted to J) if $(\nabla_x Y)(\sigma(t) = 0)$ for all $t \in J$.
The curve σ is a geodesic if $(\nabla_x X)(\sigma(t)) = 0$ for all $t \in J$. (ARTHUR A. Sagle, Ralp he, walde, 1973).

6. RIEMANNIAN METRIC

Definition (6.1) :A Riemannian metric g on amainfold M is a smooth positive definite symmetric 2-tensor field on M. This assigns to each point p ∈ M a positive definite symmetric bilinear form or inner product on the tangent space $\mathrm{T(m)_p}$

$$g_p = T(m)_p \ \times \ T(m)_p \to R$$

Recall that positive definiteness means $g_p(v, v) > 0$ for all non-zero $v \in T(m)_p$.A Riemannian manifold is a manifold with a Riemannian metric on it.The length of a tangent vector$v \in T(m)_p$ is then defined in the usual way as

$$\|v\| = g_p(v, v)^{\frac{1}{2}}$$

In terms of local coordinate system $(x_1, \dots, x_n)$ in M with basic vector fields $\delta_i = \frac{\partial}{\partial x_i}$ the local representation of g is given by

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx_i \ . \ dx_j$$

Where $g_{ij} = g(\delta_i, \delta_j)$ are real valued function on the coordinate neighborhood $U$of system. (I.M singer – J.A. Thorpe 1967).

Example (6.2):

The Euclidean space R^n with coordinates $(u_1, \dots, u_n)$ has a natural Riemannian metric

$g = \sum_{i,j=1}^{n} \delta_{ij}\, du_i \;.\; du_j = \sum_{i=1}^{n} (du_i)^2$ (Amiyamuk, 2005).

7. CONFORMAL GEOMETRY OF SUBMANIFOLD

Definition (7.1): Let N be an$M$-manifold then a subset M of N is called a submanifold of dimension$n$ if for each point $P \in M$ there is acoordinate chart $(U, \emptyset)$ at P in N such that $\emptyset$ maps $M \cap U$ homeomorphically onto an open subset of $R^u \subset R^m$where R^u is considered as subspace of the first n coordinates in R^m.

$R^n = \{(x_1, \dots, x_m) \in R^m | x_{n+1} = \cdots = x_m = 0\}$

Then the collection$\{(m \cap U, \emptyset \,| m \cap U) | (U, \emptyset)$is a chart in N, $M \cap U \neq \emptyset\}$ is a smooth atlas of m (Bohme *et al.,* 1988).

Lemma (7.2):

Let M and N be manifolds of dimensions n and $m$respectively.

If M is a submanifold of N, then for each point $P \in M$ there is on open neighbourhood U of P in N and asubmersion $g: U \longrightarrow R^{m-n}$ such that $g^{-1}(0) = M \cap U$.

Proof:

By the above definition, there is a coordinate chart $\emptyset: U \longrightarrow R^m$ a bout P in N such that if $R^m = R^n \times R^{m-n}$ then $\emptyset^{-1}(R^n \times \{0\}) = M \cap U$. Then $g = \pi o \emptyset$ where$\Pi: R^m \times R^{m-n}$is the projection on to the second factor, is a submersion with $g^{-1}(0) = M \cap U$ (Amiyamuk, 2005).

Proposition (7.3): Let M be an M-dimensional c^∞ $-$Submanifold of the *n*-dimensional C^∞ $-$manifold N and let $p \in M$. Then there exists a coordinate system (v, z) of N with $p \in v$ such that:

1. $z_1(p) = \cdots = z_n(p) = 0$ where the z_i are the coordinate functions;
2. The set $w = \{r \in v: z_{m+1}(r) = \cdots = z_n(r) = 0\}$ together with the restriction of $z_1, \dots, z_m$ to w form a chart of $M$with $p \in w$.

Conversely, if a subset $M \subset N$has a manifold structure with a coordinate system at each $P \in M$ satisfying the above then M is a submanifold of N.

Proof:

Let $F: Q \to N$ be an embedding which defines $M = F(Q)$ and let $P = F(q)$ for a unique $q \in Q$. Now let (T, y) be a chart for P in N we can assume $y(p) = 0$ in R^n.

Let U be anieghborhood of $g = f^{-1}(p)$ in Q and Let $x = yoF|U$ be such that (U, x) is a chart for $q$in Q. Thus $x(q) \in R^m$ and for $i = 1, \dots, m$ we have $x_i = y_i\ oF|U$ are the corresponding coordinate functions.

Now the composition yo fo $x^{-1} = F$ defines a c^∞ $-$Function $F: x(U) \to y(T)$, where $x(U) \in R^m$ and $y(T) \subset R^n$ and we can write F interms of coordinates $y_i = F_i(x_1, \dots x_m)$ for $i = 1, \dots, n$ the hypotheses $M = F(Q)$ is submanifold $yof = FOX$ and $x = yof|U$ yield $y_i = x_i$ for $i = 1, \dots, m$ in the above expression for F. (Sheldon w. Davis 2005).

Definition (7.4): Let $k, m \in \mathrm{N}_0$, A subset $M \subset R^k$ is called a smooth m-dinmesional submanifold of R^k if every point $P \in M$ has an open neighborhood $U \subset R^k$ such that $U \cap M$ is diffeomorphic to an open subset $\Omega \in R^m$. Adiffeomorphism$\emptyset: U \cap M \to \Omega$ is called a coordinate chart of M and its inverse $\psi = \emptyset^{-1}: \Omega \to U \cap M$ is called a(smooth) parameterization of $U \cap M$. (Abe, k, 1913).

Lemma (7.5):

If $M \subset R^k$ is a nonempty smooth *m*-manifold then $M \leq k$.

Proof:

Let $\emptyset: U \cap M \to \Omega$ be a coordinate chart of M on to an open subset $\Omega \subset R^m$, denote its inverse by $\psi = \emptyset^{-1}: \Omega \to U \cap M$and Let $P \in U \cap M$ shrinking U, if necessary, we may assume that $\emptyset$ extends to a smooth map $\Phi: U \to R^m$.

This extension satisfies $\Phi(\psi, x) = \emptyset(\psi, x) = x$ and, by the chain rule we have:

$d\Phi\big(\psi(x) d\psi(x)\big) = id: R^m \to R^m$

For every $x \in \Omega$. Hence $d\psi(x): R^m \to R^k$ is injective for $x \in \Omega$and $\Omega \neq \emptyset$ this implies $m \leq k$. (P.M. cohn and G.E.H. Reuter 1970).

Examples (7.6):

i. **Consider the 2-Sphere:**
ii.
$M = S^2 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1|\}$

Let $U \in R^3$ and $\Omega \subset R^2$be the open sets
$U = \{(x, y, z) \in R^3 / z > 0\}, \Omega = \{(x, y) \in R^2 / x^2 + y^2 < 1\}$

The map $\emptyset : U \cap M \to \Omega$ given by
$\emptyset(x, y, z) = (x, y)$

subjective and its inverse $\psi = \emptyset^{-1}: \Omega \to U \cap M$ is given by
$\psi(x, y) = \left(x, y, \sqrt{1 - x^2 - y^2}\right)$

Since both $\emptyset$ and ψ are smooth, the map $\emptyset$ is acoordinate chart os s^2.

2) Let $\Omega \subset R^m$ be an open set and $h: \Omega \to R^{k-m}$ be smooth map.

Then the graph of h is a smooth submanifold of $R^m \times R^{k-m} = R^k$.

$M = graph(h) = \{(x, y): x \in \Omega, y = h(x)\}$

It can be covered by a single coordinate chart

$\emptyset : U \cap M \to v$ where$U \coloneqq \Omega \times R^{k-m}, \emptyset$ is the projection on to n, and $\psi = \emptyset^{-1} : \Omega \to U$ is given by $\psi(x) = (x, h(x))$. (Joel w.Robbin 1976)

Definition (7.7): A Riemannian n-manifold M is called conformally flat, if at each point $x \in M$ there is a neighborhood of x in M which is conformal to the Euclidean n-space. An immersed submanifold $F: M \to E^m$ is called a conformally flat submainfold if the submanifold is conformally flat with respect to the induced metric.

Since every Riemannian 2-manifold is conformally flat, due to the existence of local isothermal coordinate system, we only consider conformally flat manifolds of dimension greater than or equal to 3.

Definition (7.8):

i. Quasi-umbilicity of conformally flat hyper surfaces the study of non flat confomally flat hyper surface of dimension $n \geq 4$ was initiated by E`. Cartan around 1918. He proved that a hyper suface of dimension > 4 in Euclidean space is conformally flat if and only if it is quasi – umbilical, that is, it has aprincipal curvature with multiplicity $\geq n - 1$.
ii. **Canal Hyper surfaces:** A hyper surfacse of Euclidean space is called a canal hyper surface if it is the envelope of one – parameter family of hyper surface.

Conformally flat hyper surfaces as Loci of spheres (4-4-3) A confomally flat hyper sufaces of dimension ≥ 4 in areal space form $R^{n+1}(c)$ is a locus of $(n - 1)$ spheres in the sense that it is given by smooth gluing of some n-dimensional submanifold of M such that each of the submanifolds is foliated by totally umbilical $(n - 1)$ submanifolds of $R^{n+1}(c)$ (Bohme *et al.,* 1988). Intrinsic properties of conformally flat hyper surfaces an intrinsic characterization of conformally flat manifold admitting isometric immersions in real space forms as hypersurfaces was given by chen and yano. A conformably flat manifold M of dimension $n \geq 4$ is called special if there exist three functions α, β and y on M such that the tensor L defined

$$L(x, y) = -\left(\frac{1}{n - 2}\right) Ric(x, y) + \left(\frac{p}{2(n - 1)(n - 2)}\right) g(x, y)$$

Where Ric is Ricci tensor and $p \equiv$ trance Ric takes the form:
$$L = -\frac{1}{2}(k + \alpha^2)g - \alpha\beta w(x)w$$

For some constant k where w is unit 1-form satisfying $d\alpha = \gamma w$ on the open subset where $L = -\frac{1}{2}(k + \alpha^2)g$
$U = \{x \in M: \beta(x) \neq 0\}\ on\ \{x \in M: \beta(x) = 0\}$

For a special conformally flat space M, we define a real number im by

$$im = \sup\left\{\begin{matrix} k \in R: L = \dfrac{k+\alpha^2}{2} g - \alpha\beta w \otimes w \\ for\ some\ funtion\ \alpha, \beta\ on\ M \end{matrix}\right\}$$

Which is called the index of the special conformably flat manifold (Bohme *et al.,* 1988).

8. SOME APPLICATION

Definition (8.1): Let X, Y be a topological spaces and $F, g: X \to Y$continuous maps. Ahomotopy from F to g is a continuous function $F: X \times [0,1] \to Y$satisfying $F(x,0) = F(x) = (x,1) = g(x)$.

Let M^m, v^{m+q} be (finite) Poincare complexes. Then an embedding of M in v shall consist of : $a(q-1)$ −spherical fibration ∈ with projection $P: E \to M$, a(finite) Poincare pair(C, E) and a (simple) homotopy equivalence $h: c \cup M(p) \to v$ where $m(p)$ is the mapping cylinder of p and $C \cup M(p) = E$ (Takahashi and Sasakian 1969).

Theorem (8.2):

Supposev^{m+q} a PL or smooth closed manifold, M^m a PL or smooth submanifold and (in the PL case) that is Locally flat in v. Then the embedding determines a finite Poincare embedding of M in v. (P.M. cohn and G.E.H. Reuter 1970) .

9. SOME APPLICATIONS TO KINEMATIC QUANTITIES OF SPACE TIMES

Threom (9.1):

Let (M, g) be a 4-dimensional space time manifold of general relativity this means that M is a smooth C^∞ connected Hausdorff 4-dimensional manifold and g is a time orientable Lorentz metric of normalhyperbolic signature $(-+++)$.

The set of all integral curves given by a unit time like or space like or null vector field u is the congruence of timelike or null curves. We first consider time like curves, also called flow lines. the acceleration of the flow lines along u is given by $\nabla_u u$ or u^a, u^b where ∇ is the Levi-Civita connection on (m, g) and $(0 \leq a, b \leq 3)$.

The projective tensor, defined by

$h_{ab} = g_{ab} + u_a u_b$
is used to project a tangent vector at a point p in the space time into a space like vector orthogonal to u at p.

The rate of change of the separation of flow lines from a time like curve, say C, tangent to u is given by the expansion tensor
$\theta_{ab} = h_a^c h_b^d u[c; d]$

The volume expansion θ the shear tensor σ_{ab}, the vorticity tensor $w_{ab}$and the vorticity vectorw^a are defined as follows:

$$\theta = \text{div u} = \theta_{\text{ab}} \text{h}_{\text{ab}}$$

$$\sigma_{ab} = \theta_{\text{ab}} - \frac{\theta}{3} \text{h}_{\text{ab}}$$

$$w_{ab} = \text{h}_a^c \text{h}_b^d \text{ u[c; d]}$$

$$w^a = \frac{1}{2} \eta^{abcd} u_b w_{cd}$$

$$\eta^{abcd} = \text{g}^{ae} \text{g}^{bf} \text{g}^{cg} \text{g}^{dh} \eta_e f_g h$$

$$\eta_e f_g h - (4!)\sqrt{-g}\, \delta^0_{[e} \delta^1_f\ \delta^2_g \delta^2_{h]},$$

Where η^{abcd} is the levi –civita valume-form. The equation above measures the rate at which the time like curves rote at about an integral curve of u (Bohme *et al.,* 1988).

Theorem (9.2):

We follow the general notations of sub manifold theory and let (m, g) be a smooth light like hyper surface of an $(m+2)$– dimensional Lorentzian manifold $(\overline{m}, \overline{g})$ we need the following consider :

$\widetilde{TM} = TM/Rad\ (TM)$
$\Pi = \Gamma(TM) - \Gamma(\widetilde{TM})$
(canonical projection)

Set $\tilde{X} = \Pi(X)$and $\tilde{g}(\tilde{x}, \tilde{y}) = g(x, y)$. It is easy to prove that the operator $\widetilde{A}u = \Gamma(\overline{TM}) \rightarrow \Gamma(\overline{TM})$ defined by $\widetilde{A}U(\widetilde{X}) = -(\Pi(\overline{\nabla}_X U))$whereU $\in$ (Rad(TM))and $x \in \Gamma(\mathrm{TM})$ is self –adjoint.

It is known that all Riemannian self. Adjointope rater are diagonalizable. Let $\{k_1, \ldots, k_n\}$ be eigen values.
If $\{S_{ki}, 1 \leq i \leq n\}$, is the eigen space of k_i

Then
$\widetilde{TM} = \widetilde{S_{k_1}} \perp \cdots \perp \widetilde{S_{k_n}}$

Theorem(9.3):

Let $c(p)$ be a null curve in $(\overline{M}, \bar{g})$ where $P \in I \subset R$ is a special parameter $\{\epsilon, N, w_1, w_2\}$ is pseudo – orthonormal $\{\epsilon, N, w_1, w_2\}$ is a pseudo – orthonormalfrenet frame along $c(p)$, $\epsilon$and $N$are null vectors such that $g(\epsilon, N) = 1 c = span\{\epsilon\}$ and w_1, w_2 are unit spacelike vectors.

If Nmoves a long $c$then, it generates a ruled surface given by the parameterization $((I \times R), F)$ where $F: I \times R \rightarrow \overline{M}$ is defind by $(P, u) \rightarrow F(p, u) = C(p) + uN(p)\ u \in I \subset R$ Above ruled surface is called a null scroll which we denote by s_c is clear by the above defining equation that the null scroll S_c is a time like ruled surface in $\overline{M}$.(John M. Lee, 1968).

Proposition (9.4):

A point x in a space time manifold $(\overline{M}, \bar{g})$ admits a neighborhood that can be foliated into time like photon 2-surfaces if and only if on some neighborhood of x there are two linearly independent null geodesic vector fields $\epsilon$and N such that the liebraket $[\epsilon, N]$ is a linear combination of ϵ and N. In this case, the photon 2-surfaces are the integral manifolds, say S_c, of the 2-suface spanned by ϵ and N (Takahashi and Sasakian, 1969).

Physical Application (9.5):

Let $(\overline{M}, \bar{g})$be the Einstein universe which may be described as a 3-sphere S^3 of a fixed radius r, i.e., as the boundary of a 4-dimensional ball, by the equator. Consider a non-constant map.

$$\emptyset: (\overline{M}, \bar{g}) \rightarrow S^2 \subset (M, g) \xrightarrow{i} R_1^4 \qquad (11.6.1)$$

Where S^2 is a leaf of a screen $S(T\ \overline{M})$of $\overline{M}$. A sub mapping of (11.6.1), given by:

$$\emptyset/S^2 : S^3 \rightarrow S^2 \subset \overline{M} \xrightarrow{i} R_1^4$$

is known as Hopf map, the most celebrated example of harmonic morphisms, with constant dilation $\lambda = 2$ and minimal fibers. Here we have related Hopf map with a spacetime and a globally null manifold.

Physical Model (9.6) :

Let $(\overline{M}, \bar{g})$ be a 4-dimensional Einstein Maxwell space time manifold of general relativity with a skew-symmetric tensor field $F = (F_{ab})$on $\overline{M}$ which represents the electromagnetic fields. The complex self-dual electromagnetic Ensor field F^* is defined by:

$$F_{ab}^* = F_{ab} + i\tilde{F}_{ab}, \qquad \tilde{F}_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}, \qquad i = \sqrt{-1}$$

Here, ϵ_{abcd} is the Levi-Civita tensor field.

Theorem (9.7):

The only Einstein Maxwell field which is homogeneous and has a homogeneous non-singular Maxwell field is the Betroth-Robinson solution

$$ds^2 = A^2(d\emptyset^2 + sin^2\emptyset\ d\ \emptyset^2 + d\ x^2 + \sinh^2 x\ \ dt^2)$$

for local coordinates $(t, x, \theta, \emptyset)$ and an arbitrary constant A.(Takahashi, T.Sasakian 1969).

10. RESULTS

We found that A Riemannian manifold is a manifold with A Riemannian metric on it, every Riemannian 2-manifold is conformally flat and most application of submanifold is physical applications.

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