International Journal of Science Academic Research
Vol. 02, Issue 04, pp.1304-1312, April, 2021
Available online at http://www.scienceijsar.com

ISSN: 2582-6425

Research Article

DEMONSTRATION OF THE SIX FIRSTS FORMULAS OF ADAMS-BASHFORTH FOR SOLVING INITIAL VALUE PROBLEMS

***Njipouakouyou Samuel**

Frmr senior lecturer, Faculty of Sciences, University of Dschang – Cameroon

Received 14th February 2021; **Accepted** 20th March 2021; **Published online** 15th April 2021

Abstract

This study concerns the demonstration of Adams-Bashforth formulas for solving multi step value problems and their corresponding error terms. It shows the users how these formulas have been obtained and enables them to write their own computer programs for their concrete problems instead of managing already prepared ones which probably do not fit with their works.

Keywords: Initial value problems, continuous phenomena, discrete phenomena, experimental data, equidistant nods, interpolating polynomials, integrals, estimations, error terms.

FEW RECALLS

When investigating functional relationships between two or more variables, appropriate instruments are used. The investigated phenomena are usually continuous and therefore, should be represented by continuous analytical functions. But none of the used instruments can give these functions. To reach this goal, continuous phenomena are transformed to discrete ones fixing in advance some nods x_i in the considered field and finding experimentally the value of the function at these nods, say $y_i = f(x_i)$, i = 0, 1, 2, …, n. Thus, the searched function is known at some nods and presented in tabular form, conferred Table 1.

Table 1. Form of presentation of a relationship between two variables x and y from an experimental information

x_i	x_0	x_1	….	x_n
y_i	y_0	y_1	….	y_n

Sometimes, the phenomenon can also be presented in graphical form when other special instruments are used. But the most encountered form is tabular. So having Table 1 representing $y_i=f(x_i)$, a polynomial of a given order n, $P_n(x)$, very closer to the unknown function y=f(x), could be found such that for any nod x_i, we have $y_i=f(x_i)=P_n(x_i)$.

For solving such problems, many polynomials have been elaborated. The most encountered ones are the Lagrange and the Newton interpolating polynomials noted by $P_n(x)$, n - the degree of P(x).

For n nods, x_i, i=0, 1, …, n, the Lagrange interpolating polynomial is:

$$P_n(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x) + \ldots. + f(x_n)L_n(x). \quad (1)$$

$L_i(x)$, i = 0, 1, 2, …, n, are the Lagrange polynomials given by:

$$L_i(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{i-1})(x-x_{i+1})\ldots.(x-x_n)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots.(x_i-x_n)}. \quad (2)$$

Writing (2) as:

$$\frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{i-1})(x-x_{i+1})\ldots.(x-x_n)}{(x_j-x_0)(x_j-x_1)(x_j-x_2)\ldots(x_j-x_{i-1})(x_j-x_{i+1})\ldots.(x_j-x_n)} = \prod_{\substack{i=0 \\ j\neq i}}^{n} \frac{(x-x_i)}{(x_j-x_i)} \quad (3)$$

and introducing the well-known sum symbol, $\sum$, (1) can be given the form:

$$P_n(x)=\sum_{j=0}^{n} f(x_j)L_j(x)=\sum_{j=0}^{n} f(x_j)\prod_{\substack{i=0 \\ j\neq i}}^{n} \frac{(x-x_i)}{(x_j-x_i)}. \quad (4)$$

Corresponding Author:** ***Njipouakouyou Samuel
Frmr senior lecturer, Faculty of Sciences, University of Dschang – Cameroon.

The Newton interpolating polynomial is:

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \dots + a_n(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1}). \quad \textbf{(5)}$$

Its coefficients are to be found recalling that:

$$P_n(x_i) = f(x_i)\ ,\ i = 0, 1, 2, \dots, n. \quad \textbf{(6)}$$

Introducing the divided differences and taking (5)-(6) into account, these coefficients are:

$$a_0 = f(x_0),\ \ a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = f\,[x_1, x_0], \quad \textbf{(7)}$$

$$a_2 = \frac{f(x_2) - f(x_0) - (x_2 - x_0) f\,[x_1, x_0]}{(x_2 - x_0)(x_2 - x_1)}. \quad \textbf{(8)}$$

Remarking that

$$f(x_2) - f(x_0) = f(x_2) - f(x_1) + f(x_1) - f(x_0) =$$
$$f(x_2) - f(x_1) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x_1 - x_0) =$$
$$f(x_2) - f(x_1) + f\,[x_1, x_0](x_1 - x_0), \quad \textbf{(9)}$$

we easily have

$$a_2 = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = f[x_2, x_1, x_0]. \quad \textbf{(10)}$$

Proceeding the same way, we come to the general formula:

$$a_i = f[x_i, x_{i-1}, x_{i-2}, \dots, x_2, x_1, x_0] = = \frac{f[x_i, x_{i-1}, x_{i-2}, \dots, x_2, x_1] - f[x_{i-1}, x_{i-2}, \dots, x_2, x_1, x_0]}{x_I - x_0}. \quad \textbf{(11)}$$

Thus, the Newton interpolating polynomial can be given the forms:

$$P_n(x) = a_0 + f[x_1,x_0](x-x_0) + f[x_2, x_1,x_0](x-x_0)(x-x_1) + f[x_3, x_2, x_1,x_0](x-x_0)(x-x_1)(x-x_2) + \dots +$$
$$f[x_n, x_{n-1}, x_{n-2}, \dots, x_2, x_1, x_0](x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-2})(x-x_{n-1}) =$$
$$f(x_0) + \sum_{i=1}^{n} f[x_i, x_{i-1}, x_{i-2}, \dots, x_2, x_1, x_0](x-x_0)(x-x_1)(x-x_2)\dots(x-x_{i-2})(x-x_{i-1}). \quad \textbf{(12)}$$

Approximations of the function f(x) given by (4) and (12) generate an error E(x) given by:

$$E(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n}(x - x_i), \quad \textbf{(13)}$$

where $f^{(n+1)}(\xi(x))$ is the derivative of order n+1 of f(x) and $\xi(x)$ - a point in the open interval generated by the nods x_i, i = 0, 1, 2, …, n and x. As f(x) and the exact position of $\xi(x)$ are undetermined, it is not possible to express the (n+1)th derivative of f(x) or to find its value. To overcome this difficulty, this derivative is generally replaced by the absolute value of the upper boundary of $f^{(n+1)}(x)$ in the considered interval. Formula (13) is related to the Lagrange interpolating polynomial. For the Newton interpolating polynomial, this error is:

$$E(x) = f(x) - P_n(x) = f[x, x_n, x_{n-1}, x_{n-2}, \dots, x_2, x_1, x_0) \prod_{i=0}^{n}(x - x_i). \quad \textbf{(14)}$$

POSITION OF THE PROBLEM

Many physical phenomena are described by the first order differential equation associated to the initial condition:

$$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0. \quad \textbf{(15)}$$

Integrating (15) in the interval $[x_i, x_{i+1}]$ gives:

$$\int_{x_i}^{x_{i+1}} \frac{dy}{dx}\,dx = \int_{x_i}^{x_{i+1}} f(x,y)dx. \quad \textbf{(16)}$$

The solution of (15) is:

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x,y)dx. \quad \textbf{(17)}$$

Assume that f(x,y) is known only at some equidistant nods, x_i, with step $h = x_i - x_{i-n}$, n = 1,…, k:
$(x_i, f(x_i,y(x_i)))$, $(x_{i-1},f(x_{i-1},y(x_{i-1})))$, $(x_{i-2}, f(x_{i-2},y(x_{i-2})))$ …,
$(x_{i-k},f(x_{i-k},y(x_{i-k})))$.

Expressing the nods in this manner indicates that we are going from the bottom to the top of the table of experimental data.
f(x,y) is unknown, so (17) cannot be determined. Thus, f(x,y) should be replaced by $P_n(x)$ defined by (4) or (12). Because of its simplicity and shorter time of execution, formula (12) is frequently used.

When starting interpolation from the top of the table of experimental data, (12) has the form:

$$P_n(x) = f(x_0) + \frac{\Delta f(x_0)}{h}(x-x_0) + \frac{\Delta^2 f(x_0)}{2!h^2}(x-x_0)(x-x_1) + \frac{\Delta^3 f(x_0)}{3!h^3}(x-x_0)(x-x_1)(x-x_2) + \dots + \frac{\Delta^n f(x_0)}{n!h^n}(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1}) \quad \textbf{(18)}$$

Where

$$\Delta f(x) = f(x+h) - f(x),$$
$$\Delta(\alpha f(x) + \beta g(x)) = \alpha\Delta f(x) + \beta\Delta g(x), \quad \textbf{(19)}$$

$$\Delta^{i+1} f(x) = \Delta(\Delta^i f(x)), \ i= 0, 1, 2, \dots$$

When starting from the bottom of the table of experimental data, (12) has the form:

$$P_n(x) = f(x_m) + \frac{\nabla f(x_m)}{h}(x-x_m) + \frac{\nabla^2 f(x_m)}{2!h^2}(x-x_m)(x-x_{m-1}) + \frac{\nabla^3 f(x_m)}{3!h^3}(x-x_m)(x-x_{m-1})(x-x_{m-2}) + \dots + \frac{\nabla^n f(x_0)}{n!h^n}(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1}) \quad \textbf{(20)}$$

Where

$$\nabla f(x) = f(x) - f(x-h),$$
$$\nabla(\alpha f(x) + \beta g(x)) = \alpha\nabla f(x) + \beta\nabla g(x), \quad \textbf{(21)}$$
$$\nabla^{i+1} f(x) = \nabla(\nabla^i f(x)), \ i= 0, 1, 2, \dots$$

(18) and (20) are respectively the forward and backward interpolating formulas of Newton.

Based on (14), (20) and (21), the interpolating polynomials of Newton for k from 0 to 5 with their error terms to replace f(x,y) are the next expressions.

For k = 0, i.e. only one interpolating nod $(x_i, f(x_i,y_i))$.

f(x,y) is estimated by:

$$P_0(x,y) = f(x_i,y_i) = f(x_i,y(x_i)), \quad \textbf{(22)}$$

and the function is:

$$f(x,y) = f(x_i,y_i) + \frac{x-x_i}{1!}f'(\eta_i,y(\eta_i)), \ x_i < \eta_i < x_{i+1}. \quad \textbf{(23)}$$

For k = 1, i.e. two nods $(x_i,f(x_i,y_i))$ and $(x_{i-1},f(x_{i-1},y_{i-1}))$.

The polynomial to estimate f(x,y) is:

$$P_1(x, y) = f(x_i,y_i) + \frac{\nabla f(x_i,y_i)}{1!h}(x-x_i), \quad \textbf{(24)}$$

and the function is:

$$f(x,y) = f(x_i,y_i)) + \frac{\nabla f(x_i,y_i)}{h}\frac{x-x_i}{1!} + f''(\eta_i,y(\eta_i))\frac{(x-x_i)(x-x_{i-1})}{2!}, \ x_i < \eta_i < x_{i+1}. \quad \textbf{(25)}$$

For k = 2, i.e. three nods $(x_i,f(x_i,y_i))$,$(x_{i-1},f(x_{i-1},y_{i-1}))$ and $(x_{i-2},f(x_{i-2},y_{i-2}))$.

The polynomial to estimate f(x,y) is:

$$P_2(x, y) = f(x_i,y_i) + \frac{\nabla f(x_i,y_i)}{1!h}(x-x_i) + \frac{\nabla^2 f(x_i,y_i)}{2!h^2}(x-x_i)(x-x_{i-1}) \quad (26)$$

and the function is:

$$f(x,y) = f(x_i,y_i) + \frac{\nabla f(x_i,y_i)}{1!h}(x-x_i) + \frac{\nabla^2 f(x_i,y_i)}{2!h^2}(x-x_i)(x-x_{i-1}) + f^{(3)}(\eta_i,y(\eta_i))\frac{(x-x_i)(x-x_{i-1})(x-x_{i-2})}{3!}, \ x_i < \eta_i < x_{i+1}. \quad (27)$$

For k = 3, i.e. four nods $(x_i,f(x_i,y_i))$, $(x_{i-1},f(x_{i-1},y_{i-1}))$, $(x_{i-2},f(x_{i-2},y_{i-2}))$ and $(x_{i-3},f(x_{i-3},y_{i-3}))$.

The polynomial to estimate f(x,y) is:

$$P_3(x, y) = f(x_i,y_i) + \frac{\nabla f(x_i,y_i)}{1!h}(x-x_i) + \frac{\nabla^2 f(x_i,y_i)}{2!h^2}(x-x_i)(x-x_{i-1}) + \frac{\nabla^3 f(x_i,y_i)}{3!h^3}(x-x_i)(x-x_{i-1})(x-x_{i-2}), \quad (28)$$

and the function is:

$$f(x,y) = f(x_i,y_i) + \frac{\nabla f(x_i,y_i)}{1!h}(x-x_i) + \frac{\nabla^2 f(x_i,y_i)}{2!h^2}(x-x_i)(x-x_{i-1}) + \frac{\nabla^3 f(x_i,y_i)}{3!h^3}(x-x_i)(x-x_{i-1})(x-x_{i-2}) + f^{(4)}((\eta_i,y(\eta_i))\frac{(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_3)}{4!},$$
$$x_i < \eta_i < x_{i+1}. \quad (29)$$

For k = 4, i.e. five nods $(x_i,f(x_i,y_i))$, $(x_{i-1},f(x_{i-1},y_{i-1}))$, $(x_{i-2}, f(x_{i-2},y_{i-2}))$, $(x_{i-3}, f(x_{i-3},y_{i-3}))$ and $(x_{i-4},f(x_{i-4},y_{i-4}))$.

The polynomial to estimate f(x,y) is:

$$P_4(x, y) = f(x_i,y_i) + \frac{\nabla f(x_i,y_i)}{1!h}(x-x_i) + \frac{\nabla^2 f(x_i,y_i)}{2!h^2}(x-x_i)(x-x_{i-1}) + \frac{\nabla^3 f(x_i,y_i)}{3!h^3}(x-x_i)(x-x_{i-1})(x-x_{i-2}) + \frac{\nabla^4 f(x_i,y_i)}{4!h^4}(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3}) \quad (30)$$

and the function is:

$$f(x,y) = f(x_i,y_i) + \frac{\nabla f(x_i,y_i)}{1!h}(x-x_i) + \frac{\nabla^2 f(x_i,y_i)}{2!h^2}(x-x_i)(x-x_{i-1}) + \frac{\nabla^3 f(x_i,y_i)}{3!h^3}(x-x_i)(x-x_{i-1})(x-x_{i-2}) + \frac{\nabla^4 f(x_i,y_i)}{4!h^4}(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3}) +$$
$$f^{(5)}((\eta_i,y(\eta_i))\frac{(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_{i-4})}{5!}, \ x_i < \eta_i < x_{i+1}. \quad (31)$$

For k = 5, i.e. six nods $(x_i,f(x_i,y_i))$, $(x_{i-1},f(x_{i-1},y_{i-1}))$, $(x_{i-2}, f(x_{i-2},y_{i-2}))$, $(x_{i-3}, f(x_{i-3},y_{i-3}))$, $(x_{i-4}, f(x_{i-4},y_{i-4}))$ and $(x_{i-5}, f(x_{i-5},y_{i-5})$.

The polynomial to estimate f(x,y) is:

$$P_5(x, y) = f(x_i,y_i) + \frac{\nabla f(x_i,y_i)}{1!h}(x-x_i) + \frac{\nabla^2 f(x_i,y_i)}{2!h^2}(x-x_i)(x-x_{i-1}) + \frac{\nabla^3 f(x_i,y_i)}{3!h^3}(x-x_i)(x-x_{i-1})(x-x_{i-2}) + \frac{\nabla^4 f(x_i,y_i)}{4!h^4}(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3}) +$$
$$\frac{\nabla^5 f(x_i,y_i)}{5!h^5}(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_{i-4}), \quad (32)$$

and the function is:

$$f(x,y) = f(x_i,y_i) + \frac{\nabla f(x_i,y_i)}{1!h}(x-x_i) + \frac{\nabla^2 f(x_i,y_i)}{2!h^2}(x-x_i)(x-x_{i-1}) + \frac{\nabla^3 f(x_i,y_i)}{3!h^3}(x-x_i)(x-x_{i-1})(x-x_{i-2}) + \frac{\nabla^4 f(x_i,y_i)}{4!h^4}(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3}) +$$
$$\frac{\nabla^5 f(x_i,y_i)}{5!h^5}(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_{i-4}) + f^{(6)}((\eta_i,y(\eta_i))\frac{(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_{i-4})(x-x_{i-5})}{6!}, \ x_i < \eta_i < x_{i+1}. \quad (33)$$

The complexity of the interpolating formulas gradually increases with k. This is why higher order interpolating polynomials are rarely encountered. When going through the literature, only final operational formulas for solving (15) and the orders of their corresponding error terms generated when replacing f(x,y) by $P_n(x,y)$ are given. The goal of the present work is to demonstrate the six firsts formulas of Adamas-Bashforth, k = 0, 1;2, 3, 4, 5, with their corresponding error terms. Surely that this will permit a deeply understanding of these formulas and will enable the users and researchers to develop fitting formulas for solving other specific problems, appreciating at the same time terms to be neglected and the errors generated.

DEMONSTRATION OF THE SIX FIRSTS ADAMS-BASHFORTH FORMULAS

Recall the solution (17) of (15) and (33). We have the integral:

$$\int_{x_i}^{x_{i+1}} f(x,y)dx = \int_{x_i}^{x_{i+1}} (f(x_i,y_i) + \frac{\nabla f(x_i,y_i)}{1!h}(x-x_i) + \frac{\nabla^2 f(x_i,y_i)}{2!h^2}(x-x_i)(x-x_{i-1}) + \frac{\nabla^3 f(x_i,y_i)}{3!h^3}(x-x_i)(x-x_{i-1})(x-x_{i-2}) +$$

$\frac{\nabla^4 f(x_i,y_i)}{4!h^4}(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3}) + \frac{\nabla^5 f(x_i,y_i)}{5!h^5}(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_{i-4}) +$
$f^{(6)}((\eta_i,y(\eta_i))\frac{(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_{i-4})(x-x_{i-5})}{6!})\mathbf{dx}.$ **(34)**

The second member of (34) contains seven integrals which should be calculated step by step.
First integral:

$$\int_{x_i}^{x_{i+1}} f(x_i,y_i)dx=(f(x_i,y_i)=const)=f(x_i,y_i)\int_{x_i}^{x_{i+1}} dx=f(x_i,y_i)(x_{i+1}-x_i)= f(x_i,y_i)h. \quad (35)$$

Recall that the interpolating nods are equidistant.
Second integral:

$$\int_{x_i}^{x_{i+1}} (\frac{\nabla f(x_i,y_i)}{1!h}(x-x_i))dx = \frac{\nabla f(x_i,y_i)}{1!h}\int_{x_i}^{x_{i+1}} (x-x_i)dx = \frac{\nabla f(x_i,y_i)}{1!h}\frac{(x_{i+1}-x_i)^2}{2}$$
$$=\frac{f(x_i,y_i)-f(x_{i-1},y_{i-1})}{h}\frac{h^2}{2} = (f(x_i,y_i)-f(x_{i-1},y_{i-1}))\frac{h}{2}. \quad (36)$$

Third integral:

$$\int_{x_i}^{x_{i+1}} (\frac{\nabla^2 f(x_i,y_i)}{2!h^2}(x-x_i)(x-x_{i-1})dx = \frac{\nabla^2 f(x_i,y_i)}{2!h^2}\int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})dx.$$
(37)

Recalls:

$$\frac{\nabla^2 f(x_i)}{2!h^2} = f[x_{i-2}, x_{i-1}, x_i] = \frac{f[x_{i-2},x_{i-1}]- f[x_{i-1},x_i]}{x_{i-2}- x_i} = \frac{f[x_i,x_{i-1}]- f[x_{i-1},x_{i-2}]}{x_i- x_{i-2}} = \frac{\frac{f(x_i)- f(x_{i-1})}{h}-\frac{f(x_{i-1})- f(x_{i-2})}{h}}{2h} = \frac{f(x_i)}{2h^2} - 2\frac{f(x_{i-1})}{2h^2} + \frac{f(x_{i-2})}{2h^2} \quad (38)$$

Considering (38), the coefficient of (37) is:

$$\frac{\nabla^2 f(x_i,y_i)}{2!h^2} = \frac{f(x_i,y_i)}{2h^2} - 2\frac{f(x_{i-1},y_{i-1})}{2h^2} + \frac{f(x_{i-2},y_{i-2})}{2h^2} \quad (39)$$

As the nods are equidistant, they can be defined by the formulas:

$x = x_i + ht$ whence $dx = hdt$, $x_{i-1} = x_i - h$, $(x-x_i)(x-x_{i-1})dx = (x_i + ht - x_i)(x_i + ht - x_i + h)hdt = h^3(t^2 + t)dt$ **(40)**

For $x = x_i$, $t = 0$ and for $x = x_{i+1}$, $t = 1$ so the integral in (37) gives:

$$\int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})dx = h^3\int_{0_i}^{1} (t^2 + t)dt = \frac{5h^3}{6} \quad (41)$$

Whence the searched integral:

$$\frac{\nabla^2 f(x_i,y_i)}{2!h^2}\int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})dx = (\frac{f(x_i,y_i)}{2h^2} - 2\frac{f(x_{i-1},y_{i-1})}{2h^2} + \frac{f(x_{i-2},y_{i-2})}{2h^2})\frac{5h^3}{6} = \frac{h}{12}(5\frac{f(x_i,y_i)}{2h^2} - 10\frac{f(x_{i-1},y_{i-1})}{2h^2} + 5\frac{f(x_{i-2},y_{i-2})}{2h^2}) \quad (42)$$

Fourth integral:

$$\frac{\nabla^3 f(x_i,y_i)}{3!h^3}\int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})(x-x_{i-2})dx \quad (43)$$

Considering (40), the integral of (43) becomes:

$$\int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})(x-x_{i-2})dx = h^4 \int_{0_i}^{1} (t^3 + 3t^2 + 2t)dt = \frac{9h^4}{4} \tag{44}$$

Recalls:

$$f[x_i, x_{i-1}, x_{i-2}, x_{i-3}] = \frac{f[x_i,x_{i-1},x_{i-2}] - f[x_{i-1},x_{i-2},x_{i-3}]}{x_i - x_{i-3}}$$
$$= \frac{f[x_i,x_{i-1},x_{i-2}] - f[x_{i-1},x_{i-2},x_{i-3}]}{3h}$$
$$= \frac{\frac{f[x_i,x_{i-1}] - f[x_{i-1},x_{i-2}]}{2h} - \frac{f[x_{i-1},x_{i-2}] - f[x_{i-2},x_{i-3}]}{2h}}{3h}$$
$$= \frac{f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})}{6h^3} = \frac{\nabla^3 f(x_i)}{3!h^3}. \tag{45}$$

So the coefficient of (43) is:

$$\frac{\nabla^3 f(x_i,y_i)}{3!h^3} = \frac{f(x_i,y_i) - 3f(x_{i-1}y_{i-1}) + 3f(x_{i-2},y_{i-2}) - f(x_{i-3},y_{i-3})}{6h^3} \tag{45a}$$

and the fourth integral is:

$$\frac{\nabla^3 f(x_i,y_i)}{3!h^3} \int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})(x-x_{i-2})dx =$$
$$\frac{9h^4}{4} \frac{f(x_i,y_i) - 3f(x_{i-1}y_{i-1}) + 3f(x_{i-2},y_{i-2}) - f(x_{i-3},y_{i-3})}{6h^3} =$$
$$\frac{9h}{24}(f(x_i, y_i) - 3 f(x_{i-1}, y_{i-1}) + 3 f(x_{i-2}, y_{i-2}) - f(x_{i-3}, y_{i-3})). \tag{46}$$

Fifth integral:

$$\frac{\nabla^4 f(x_i,y_i)}{4!h^4} \int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})dx \tag{47}$$

Considering (40), the integral of (47) is:

$$\int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})\, dx =$$
$$h^5 \int_0^1 (t^4+6t^3+11t^2+6t)dt = \frac{251}{30}h^5 \tag{48}$$

Recalls:

$$f[x_i, x_{i-1}, x_{i-2}, x_{i-3}, x_{i-4}] = \frac{f[x_i,x_{i-1},x_{i-2},x_{i-3}] - f[x_{i-1},x_{i-2},x_{i-3},x_{i-4}]}{x_i - x_{i-4}} = (x_i - x_{i-4} = 4h) = \frac{\frac{f[x_i,x_{i-1},x_{i-2}] - f[x_{i-1},x_{i-2},x_{i-3}]}{x_i - x_{i-3}} - \frac{f[x_{i-1},x_{i-2},x_{i-3}] - f[x_{i-2},x_{i-3},x_{i-4}]}{x_{i-1} - x_{i-4}}}{4h}$$

$= (x_i - x_{i-3} = 3h,\ x_{i-1} - x_{i-4} = 3h) =$ (proceed step by step by analogy) =

$$= \frac{1}{4!h^4}(f(x_i) - 4f(x_{i-1}) + 6f(x_{i-2}) - 4f(x_{i-3}) + f(x_{i-4})) = \frac{\nabla^4 f(x_i)}{4!h^4}$$

whence the searched coefficient and integral, respectively:

$$\frac{\nabla^4 f(x_i,y_i)}{4!h^4} == \frac{1}{4!h^4} f(x_i,y_i) - 4f(x_{i-1},y_{i-1}) + 6f(x_{i-2},y_{i-2}) - 4f(x_{i-3},y_{i-3}) + f(x_{i-4},y_{i-4})), \tag{49}$$

$$\frac{\nabla^4 f(x_i,y_i)}{4!h^4} \int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})\, dx. = \frac{251h}{720}(f(x_i,y_i) - 4f(x_{i-1},y_{i-1}) + 6f(x_{i-2},y_{i-2}) - 4f(x_{i-3},y_{i-3}) + f(x_{i-4},y_{i-4}))) \tag{50}$$

Sixth integral:

$$\frac{\nabla^5 f(x_i,y_i)}{5!h^5} \int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_4)dx \tag{51}$$

Considering (40), the integral of (51) is:

$$\int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_{-4})dx = h^6\int_0^1 (t^5+10t^4+35t^3+50t^2+24t)dt = \frac{475}{12}h^6 \quad \textbf{(52)}$$

Recalls:

$f[x_i, x_{i-1}, x_{i-2}, x_{i-3}, x_{i-4}, x_{i-5}]= \frac{f[x_i,x_{i-1},x_{i-2},x_{i-3},x_{i-4}]-f[x_{i-1},x_{i-2},x_{i-3},x_{i-4},x_{i-5}]}{x_i-x_{i-5}} = (x_i-x_{i-5} = 5h) =$
= (confer to the previous coefficients) =
$\frac{1}{120h^5}(f(x_i)-5f(x_{i-1})+10f(x_{i-2})-10f(x_{i-3})+5f(x_{i-4})-f(x_{i-5})) = \frac{\nabla^5 f(x_i)}{5!h^5}$.

Consequently the searched coefficient and corresponding integral are respectively:

$$\frac{\nabla^5 f(x_i,y_i)}{5!h^5} = \frac{1}{120h^5}(f(x_i,y_i)-5f(x_{i-1},y_{i-1})+10f(x_{i-2},y_{i-2}) -10f(x_{i-3},y_{i-3})+5f(x_{i-4},y_{i-4})-f(x_{i-5},y_{i-5})), \quad \textbf{(53)}$$

$$\frac{\nabla^5 f(x_i,y_i)}{5!h^5}\int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-4})dx = \frac{475h}{1440}(f(x_i,y_i)-5f(x_{i-1},y_{i-1})+10f(x_{i-2},y_{i-2})-10f(x_{i-3},y_{i-3})+5f(x_{i-4},y_{i-4}) -f(x_{i-5},y_{i-5})) \quad \textbf{(54)}$$

From the expressions of the above six firsts integrals of (37), it is possible to obtain different estimations of the solution of (15) for different k called the formulas of Adams-Bashforth. Moreover, it is also possible to develop the expression of the error terms generated by these estimations, assuming that f(x,y) is continue with its derivatives till needed order.

For k=0, (35) is the estimation of the solution of (15):

$$y(x_{i+1})=y(x_i)+hf(x_i,y_i), \quad \textbf{(55)}$$

and its error term is given by the integral of the last term of (23):

$$\int_{x_i}^{x_{i+1}} \frac{x-x_i}{1!} f'(\eta_i,y(\eta_i)) = f'(\xi_i,y(\xi_i))\int_{x_i}^{x_{i+1}} (x-x_i)dx = f'(\xi_i,y(\xi_i))\frac{h^2}{2},\ x_i < \eta_i, \xi_i < x_{i+1}. \quad \textbf{(56)}$$

Adding (55) and (56) gives the exact solution of (15):

$$y(x_{i+1})=y(x_i)+hf(x_i,y_i) + f'(\xi_i,y(\xi_i))\frac{h^2}{2}, \quad \textbf{(57)}$$

which is usually called the one step formula of Euler.

For k=1, add (36) and (55) to obtain the estimation of the solution of (15):

$$y(x_{i+1})=y(x_i)+hf(x_i,y_i)+ (f(x_i,y_i)-f(x_{i-1},y_{i-1}))\frac{h}{2} = = y(x_i)+ (3f(x_i,y_i)-f(x_{i-1},y_{i-1}))\frac{h}{2}, \quad \textbf{(58)}$$

which the Adams-Bashforth formula of order two.

Its error term is given by the integral, conferred (25):

$$\int_{x_i}^{x_{i+1}} f''(\eta_i,y(\eta_i))\frac{(x-x_i)(x-x_{i-1})}{2!}dx = \frac{1}{2!}f''(\xi_i,y(\xi_i))\int_{x_i}^{x_{i+1}} (x-x_i)(x-x_{i-1})dx=$$

$$\frac{1}{2!}f''(\xi_i,y(\xi_i))h^3\int_0^1 t(t+1)dx = \frac{5}{12}h^3f''(\xi_i,y(\xi_i)),\ x_i < \eta_i, \xi i < x_{i+1}. \quad \textbf{(59)}$$

Whence the exact solution of (15) adding (59) and (58):

$$y(x_{i+1}) = y(x_i) + (3f(x_i,y_i)-f(x_{i-1},y_{i-1}))\frac{h}{2} + \frac{5}{12}h^3f''(\xi_i,y(\xi_i)),\ x_i < \eta_i, \xi i < x_{i+1} \quad \textbf{(60)}$$

For k=2, adding (58) and (42) gives the estimation of the solution of (15):

$$y(x_{i+1}) = y(x_i) + (3f(x_i,y_i) - f(x_{i-1},y_{i-1}))\frac{h}{2} + \frac{h}{12}\left(5\frac{f(x_i,y_i)}{2h^2} - 10\frac{f(x_{i-1},y_{i-1})}{2h^2} + 5\frac{f(x_{i-2},y_{i-2})}{2h^2}\right) = y(x_i) + \frac{1}{12}h(23f(x_i,y_i) - 16f(x_{i-1},y_{i-1}) + 5f(x_{i-2},y_{i-2})), \quad \textbf{(61)}$$

(61) is the Adams-Bashforth formula of order three. Its error term is given by the integral, conferred (27):

$$\int_{x_i}^{x_{i+1}} f^{(3)}(\eta_i,y(\eta_i))\frac{(x-x_i)(x-x_{i-1})(x-x_{i-2})}{3!}dx =$$

$$f^{(3)}(\xi_i,y(\xi_i))\int_{x_i}^{x_{i+1}} \frac{(x-x_i)(x-x_{i-1})(x-x_{i-2})}{3!}dx = \frac{1}{3!}h^4\int_0^1 (t^3+3t^2+2t)dt = \frac{3}{8}h^4 f^{(3)}(\xi_i,y(\xi_i)),\ ,\ x_i < \eta_i, \xi_i < x_{i+1}, \quad \textbf{(62)}$$

whence the exact solution of the set problem adding (62) and (61):

$$y(x_{i+1}) = y(x_i) + \frac{1}{12}h(23f(x_i,y_i) - 16f(x_{i-1},y_{i-1}) + 5f(x_{i-2},y_{i-2})) + \frac{3}{8}h^4 f^{(3)}(\xi_i,y(\xi_i)),\ ,\ x_i < \eta_i, \xi_i < x_{i+1} \quad \textbf{(63)}$$

For k=3, add (61) and (46) to obtain the estimation of the solution:

$$y(x_{i+1}) = y(x_i) + \frac{1}{12}h(23f(x_i,y_i) - 16f(x_{i-1},y_{i-1}) + 5f(x_{i-2},y_{i-2})) + \frac{9h}{24}(f(x_i,y_i) - 3f(x_{i-1},y_{i-1}) + 3f(x_{i-2},y_{i-2}) - f(x_{i-3},y_{i-3})).$$

$$= y(x_i) + \frac{h}{24}(55f(x_i,y_i) - 59f(x_{i-1},y_{i-1}) + 37f(x_{i-2},y_{i-2})) - 9f(x_{i-3},y_{i-3})) \quad \textbf{(64)}$$

(64) is the Adams-Bashforth formula of order four. Its error term is given by the integral, conferred (29):

$$\int_{x_i}^{x_{i+1}} f^{(4)}((\eta_i,y(\eta_i))\frac{(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_3}{4!}dx =$$

$$\frac{1}{4}h^5 f^{(4)}((\xi_i,y(\xi_i))\int_0^1 t(t+1)(t+2)(t+3)dt = \frac{251}{720}h^5 f^{(4)}((\xi_i,y(\xi_i)),\ x_i < \eta_i, \xi_i < x_{i+1}, \quad \textbf{(65)}$$

whence the exact solution of (15) is given by the expression:

$$y(x_{i+1}) = y(x_i) + \frac{h}{24}(55f(x_i,y_i) - 59f(x_{i-1},y_{i-1}) + 37f(x_{i-2},y_{i-2})) - 9f(x_{i-3},y_{i-3})) + \frac{251}{720}h^5 f^{(4)}((\xi_i,y(\xi_i)),\ x_i < \eta_i, \xi_i < x_{i+1} \quad \textbf{(66)}$$

For k=4, add (64) and (50) to obtain the estimation of the solution:

$$y(x_{i+1}) = y(x_i) + \frac{h}{24}(55f(x_i,y_i) - 59f(x_{i-1},y_{i-1}) + 37f(x_{i-2},y_{i-2})) - 9f(x_{i-3},y_{i-3})) + \frac{251h}{720}(f(x_i,y_i) - 4f(x_{i-1},y_{i-1}) + 6f(x_{i-2},y_{i-2}) - 4f(x_{i-3},y_{i-3}) + f(x_{i-4},y_{i-4}))) = y(x_i) + \frac{1}{720}h(1901f(x_i,y_i) - 2774f(x_{i-1},y_{i-1}) + 2616f(x_{i-2},y_{i-2}) - 1274f(x_{i-3},y_{i-3}) + 251f(x_{i-4},y_{i-4}))) \quad \textbf{(67)}$$

which is the Adams-Bashforth formula of order five. Its error term is given by the integral, conferred (31):

$$\int_{x_i}^{x_{i+1}} f^{(5)}((\eta_i,y(\eta_i))\frac{(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_{i-4})}{5!}dx =$$

$$\frac{1}{4}h^6 f^{(5)}((\xi_i,y(\xi_i))\int_0^1 t(t+1)(t+2)(t+3)(t+4)dt = \frac{95}{288}h^6 f^{(5)}((\xi_i,y(\xi_i)),\ x_i < \eta_i, \xi_i < x_{i+1} \quad \textbf{(68)}$$

So the exact solution of the set problem is given by the expression:

$$y(x_{i+1}) = y(x_i) + \frac{1}{720}h(1901f(x_i,y_i) - 2774f(x_{i-1},y_{i-1}) + 2616f(x_{i-2},y_{i-2}) - 1274f(x_{i-3},y_{i-3}) + 251f(x_{i-4},y_{i-4}))) + \frac{95}{288}h^6 f^{(5)}((\xi_i,y(\xi_i)),\ x_i < \eta_i, \xi_i < x_{i+1} \quad \textbf{(69)}$$

For k=5, add (54) and (67) to obtain the estimation of the solution:

$$y(x_i) + \frac{1}{720}h(1901f(x_i,y_i) - 2774f(x_{i-1},y_{i-1}) + 2616f(x_{i-2},y_{i-2}) -$$

$$1274f(x_{i-3},y_{i-3}) + 251f(x_{i-4},y_{i-4}))) + \frac{475h}{1440}(f(x_i,y_i) - 5f(x_{i-1},y_{i-1}) + 10f(x_{i-2},y_{i-2}) - 10f(x_{i-3},y_{i-3}) + 5f(x_{i-4},y_{i-4}) - f(x_{i-5},y_{i-5}))$$

$$= y(x_i) + \frac{1}{12960}h(38494f(x_i,y_i) - 71307f(x_{i-1},y_{i-1}) + 89838f(x_{i-2},y_{i-2}) - 65682f(x_{i-3},y_{i-3}) + 25893f(x_{i-4},y_{i-4}) - 4275f(x_{i-5},y_{i-5})) \quad \textbf{(70)}$$

(70) is the Adams-Bashforth formula of order six. Its error term is given by the integral, conferred (33)::

$\int_{x_0}^{x_1}$ $f^{(6)}((\eta_i, y(\eta_i))\frac{(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_{i-4})(x-x_{i-5})}{6!})dx=$

$=\frac{1}{6!}f^{(6)}((\xi_i, y(\xi_i))\int_{x_0}^{x_1}$ $(x-x_i)(x-x_{i-1})(x-x_{i-2})(x-x_{i-3})(x-x_{i-4})(x-x_{i-5})dx =\frac{19087}{84}h^7 f^{(6)}((\xi_i, y(\xi_i))$, $x_i < \eta_i, \xi_i < x_{i+1}$ **(71)**

Whence the exact solution of (15) obtained adding (71) and (70):

$y(x_{i+1}) = y(x_i) + \frac{1}{12960}h(38494f(x_i,y_i) - 71307f(x_{i-1},y_{i-1}) + 89838 f(x_{i-2},y_{i-2}) - 65682 f(x_{i-3},y_{i-3}) + 25893 f(x_{i-4},y_{i-4}) - 4275 f(x_{i-5},y_{i-5})) + \frac{19087}{84}h^7 f^{(6)}((\xi_i, y(\xi_i))$, $x_i < \eta_i, \xi_i < x_{i+1}$ **(72)**

As indicated above, the expressions of these error terms are operationally unused conditions as the exact position of the nod ξ_i is unknown. To overcome this difficulty, users frequently determine the maximum absolute value of the corresponding derivative in the interval $[x_i, x_{i+1}]$ to estimate their upper boundaries.

To end this section, some domains of applicability of the established formulas should be pointed out. Obviously, these domains are various. Mining prospections and weather forecast could be pointed, between others. Remark that for i=0, formula (70), gives:

$y(x_1) = y(x_0) + \frac{1}{12960}h(38494f(x_0,y_0) - 71307f(x_{-1},y_{-1}) + 89838 f(x_{-2},y_{-2}) - 65682 f(x_{-3},y_{-3}) + 25893 f(x_{-4},y_{-4}) - 4275 f(x_{-5},y_{-5}))$. **(73)**

In the case of the weather forecast the variable x should represent the time in days and y(x) – the daily temperature of the air, $y(x_{0+1})=y(x_1)$ should be the daily temperature of the air the coming next day. Not only the temperature of the actual day, $y(x_0)$, should be needed, but also its values registered five days before should be required, respectively, $y(x_{-1})$, $y(x_{-2})$, $y(x_{-3})$, $y(x_{-4})$ and $y(x_{-5})$. So, six initial values should be needed. If these values are not available, they should be estimated by other formulas like the Runge-Kutta ones, the most encountered. When i will increase, the number of initial values will decrease to zero when $i \geq 6$. Similar analysis could be done for other estimations.

CONCLUSION

The demonstration of these formulas will certainly easy their better understanding by the users and enable them to write more complete computer programs for solving concrete problems instead of using already prepared ones written for other works probably different from what they have. Moreover, users will be able to estimate the upper boundary of the errors generated by correspondent estimations, what will help them having more accurate appreciation on their results.

REFERENCES

Melentev P.V. *Approximate computations*. Gosudarstvenoe izdatelsvo physico-mathematisheskoe literaturi, Moskva 1962, p. 192-260.

Patel V.A. *Numerical analysis*. Saunders College Publishing, Harcourt Brace College Publishers, 1994, p. 278-305.

****