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Research Article

FUZZY DOT IDEAL OF *BCK/BCI*-ALGEBRAS

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Abstract

The concept of fuzzy dot subalgebra of *BCK/BCI*-algebras was introduced by Jun and Hong [5]. In this paper, we introduce the concept of fuzzy dot ideal, and study its some characterizations and properties. Also, we give a relation between a fuzzy dot ideal in theorems.

Keywords: *BCK/BCI*-algebras, fuzzy dot subalgebra, fuzzy dot ideal.

INTRODUCTION

The notion of *BCK*-algebra was introduced by Imai and Iseki in 1966 [2]. In the same year Iseki [3] introduced the notion of a *BCI*-algebra which is a generalization of a *BCK*-algebra. After the introduction of the concept of fuzzy sets by Zadeh [11], several researches worked on the generalization of the notion of fuzzy sets. Jun and Hong [5] introduced a fuzzy dot subalgebra in *BCK/BCI*-algebras and investigated some properties. In this paper, we introduce the notion of fuzzy dot ideal and give some fundamental properties and characterizations of fuzzy dot ideal of *BCK/BCI*-algebra.

PRELIMINARIES

In this section, some basic definitions and properties of *BCK/BCI*-algebras and fuzzy sets in *BCK/BCI*-algebras are given.
By a *BCI*-algebra X , we mean an algebra $(X,*,0)$ of type $(2,0)$ satisfying the following conditions:

$BCI-1$ $((xy)(xz))(zy)=0,$

$BCI-2$ $(x(xy))y=0,$

$BCI-3$ $xx=0,$

$BCI-4$ $xy=0$ and $yx=0 \Rightarrow x=y,$

where, $xy=x*y$, and $xy=0$ if and only if $x \le y$ for all $x,y,z \in X$.

A *BCI*-algebra X satisfying $0 \le x$, for all $x \in X$ is called a *BCK-algebra*. In a *BCK/BCI*-algebra X ,the following properties hold for all $x,y,z \in X$.

P-1 $x0=x$.
P-2 $(xy)z=(xz)y$.
P-3 $x \le y$ implies that $xz \le yz$ and $zy \le zx$.
P-4 $(xz)(yz) \le xy$ [5].

If X is a *BCK*-algebra, then the inequality $xy \le x$ holds for all $x,y \in X$

Next, we review some fuzzy concepts. A fuzzy set of X is a function $\mu : X \to [0,1]$. The set $\mu_t = \{x \in X \mid \mu(x) \ge t\}$, where $t \in [0,1]$ is called the level subset of μ.

A nonempty subset I of a *BCK/BCI*-algebra X is called an *ideal* of X , if it satisfies :
(I-1) $0 \in I$,
(I-2) $xy \in I$ and $y \in I$ imply that $x \in I$, for all $x,y \in X$.

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A fuzzy set μ of a *BCK/BCI*-algebra X is said to be a *fuzzy ideal* ([1],[4]) of X, if it satisfies:
(FI-1) $\mu(0) \geq \mu(x)$,

(FI-2) $\mu(x) \geq \min\{\mu(xy), \mu(y)\}$, for all $x, y \in X$.

FUZZY DOT IDEAL

Definition 3.1. ([5],[6]) Let μ be a fuzzy set in a *BCI*-algebra X. Then μ is called a *fuzzy dot subalgebra* (also called fuzzy H-algebra [6]) of X, if it satisfies:

$\mu(xy) \geq \mu(x)\mu(y)$, for all $x, y \in X$.

Definition 3.2. Let μ be a fuzzy set in a *BCI*-algebra X. Then μ is called a *fuzzy dot ideal* of X, if it satisfies:
(FD-1) $\mu(0) \geq \mu(x)$,

(FD-2) $\mu(x) \geq \mu(xy)\mu(y)$, for all $x, y \in X$.

Example 3.3. Let $X = \{0, a, b, c\}$ be a *BCI*-algebra with $*$ defined by

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

[Example 3.2]. Define the fuzzy subset μ of X by $\mu(0) = 0.8$, $\mu(a) = \mu(b) = 0.25$ and $\mu(c) = 0.1$. Routine calculations give that μ is a fuzzy dot ideal of X.

Example 3.4. Let $X = \{0, a, b, c\}$ be a *BCK*-algebra with $*$ defined by

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

[3; Example 3.2]. Define the fuzzy subset μ of X by $\mu(0) = 0.9$, $\mu(a) = \mu(b) = 0.5$ and $\mu(c) = 0.1$. Routine calculations give that μ is a fuzzy dot ideal of X. Also a fuzzy subset ν of X, defined by $\nu(0) = \nu(a) = 0.5$, $\nu(b) = 0.4$ and $\nu(c) = 0.3$, is a fuzzy dot ideal of X.

Remark 3.5. Note that every fuzzy ideal of X is a fuzzy dot ideal of X**,** since

$$\mu(x) \geq \min\{\mu(xy), \mu(y)\} \geq \mu(xy)\mu(y)$$

but the converse is not true. In Example 3.4. we can see the fuzzy dot ideal ν is not a fuzzy ideal of X, since

$$\begin{aligned} \nu(b) = 0.4 &< \min\{\nu(ba), \nu(a)\} \\ &= \min\{\nu(a), \nu(a)\} = 0.5 \end{aligned}$$

Proposition 3.6. *Let D be a nonempty subset of a BCK/BCI-algebra X and μ_D a fuzzy set in X defined by $\mu_D(x)=s$ if $x \in D$ and $\mu_D(x)=t$ otherwise, $s,t \in [0,1]$ with $s>t$. Then μ_D is fuzzy dot ideal of X, if D is ideal of X.*

Proof. Suppose that D is an ideal of X. Since $0 \in D$, we have $\mu_D(0)=s \geq \mu_D(x)$, for all $x \in X$. Let $x,y \in X$. If $xy \in D$ and $y \in D$, then $x \in D$, so $\mu_D(x)=s \geq \mu_D(xy)\mu_D(y)=s^2$. If $xy \notin D$ or $y \notin D$, then $\mu_D(xy)\mu_D(y)=ts \leq t \leq \mu_D(x)$. If $xy \notin D$ and $y \notin D$, then $\mu_D(xy)\mu_D(y)=t^2 \leq t \leq \mu_D(x)$. Therefore μ_D is a fuzzy dot ideal of X.

Proposition 3.7. *Every fuzzy dot ideal μ of a BCK/BCI-algebra X with $\mu(0)=1$, is order reserving.*

Proof. Let $x,y \in X$. If $x \leq y$, then $xy=0$, so

$\mu(x) \geq \mu(xy)\mu(y) = \mu(0)\mu(y) = \mu(y)$.

Proposition 3.8. *Let μ be a fuzzy dot ideal of a BCK/BCI-algebra X*, and $\mu(0)=1$. *Then for all $x,y,z \in X$, it satisfies the condition (1) $\mu(xy) \geq \mu((xy)y)$, if and only if it satisfies*

(2) $\mu((xz)(yz)) \geq \mu((xy)z)$

Proof. *Let μ be a fuzzy dot ideal of X* satisfying (1). Since $((x(yz))z)z = ((xz)(yz))z \leq (xy)z$, by Proposition 3.7. we have $\mu((x(yz))z)z \geq \mu((xy)z)$. It follows from (1) that.

$$\begin{aligned}\mu((xz)(yz)) &= \mu((x(yz))z) \\ &\geq \mu(((x(yz))z)z) \\ &\geq \mu((xy)z)\end{aligned}$$

Thus μ satisfies (2).
Conversely, replacing z with y in (2), we obtain the condition (1). This completes the proof.
We denote $x(xy)=x^2y$ and inductively $x(\ldots(xy))=x^ny$, if x accuses n-time.

Proposition 3.9. *Let μ be a fuzzy dot ideal of a BCK/BCI-algebra X*, and $\mu(0)=1$. *Then for all $x,y \in X$ we have*

(i) $\mu(\mathrm{xy})^n \geq (\mu(x))^2$, *where* $n=2k$, $k \in \square$.
(ii) $\mu(\mathrm{xy})^n \geq \mu((xy)x)\mu(x)$, *where* $n=2k+1$, $k \in \square$.
(iii) $\mu(\mathrm{x}^\mathrm{n}) \geq \mu(xy)\mu(y)$, *where* $n=2k+1$, $k \in \square$.

Proof. Let $x,y \in X$, since

$$\begin{aligned}(xy)x &= (xx)y = 0y \\ (xy)^2x &= (xy)((xy)x) = (xy)(0y) \leq x0 = x \\ (xy)^3x &= (xy)((xy)^2x) \leq (xy)x = 0y \\ (xy)^4x &= (xy)((xy)^3x) \leq (xy)(0y) \leq x0 = x \\ &\vdots\end{aligned}$$

$(xy)^{2k} x \leq x$, (1)

$(xy)^{2k+1} x \leq 0y$. (2)

(i) By (1) and Proposition 3.7. we have $\mu\left((xy)^{2k} x\right) \geq \mu(x)$, then

$$\begin{aligned} \mu(xy)^{2k} &\geq \mu\left((xy)^{2k} x\right)\mu(x) \\ &\geq \mu(x)\mu(x) \\ &= (\mu(x))^2 \end{aligned}$$

(ii) By (2) and Proposition 3.7. we have $\mu\left((xy)^{2k+1} x\right) \geq \mu(0y)$, then

$$\begin{aligned} \mu\left((xy)^{2k+1}\right) &\geq \mu\left((xy)^{2k+1} x\right)\mu(x) \\ &\geq \mu(0y)\mu(x) \\ &= \mu\left((xy)x\right)\mu(x) \end{aligned}$$

(iii) Since $x^{2k+1}(xy) \leq y$, then by Proposition 3.7. we get $\mu\left(x^{2k+1}(xy)\right) \geq \mu(y)$, then

$$\begin{aligned} \mu(x^n) &\geq \mu\left(x^n(xy)\right)\mu(xy) \\ &\geq \mu(y)\mu(xy) \\ &= \mu(xy)\mu(y) \end{aligned}$$

Theorem 3.10. *Let X be a BCK/BCI-algebra, and let μ be a fuzzy set of X and $\mu(0) = 1$. Then μ is a fuzzy dot ideal of X if and only if it satisfies.*

$xy \leq z$ *implies* $\mu(x) \geq \mu(y)\mu(z)$, *for all* $x, y, z \in X$.

Proof. Suppose that μ is a fuzzy dot ideal of X . Let $xy \leq z$ for all $x, y, z \in X$. By Proposition 3.6. $\mu(xy) \geq \mu(z)$, so

$$\begin{aligned} \mu(x) &\geq \mu(xy)\mu(y) \\ &\geq \mu(z)\mu(y) \end{aligned}$$

Conversely, since $x(xy) \leq y$, then by hypothesis we get $\mu(x) \geq \mu(xy)\mu(y)$. Hence μ *is a fuzzy dot ideal of* X .

Theorem 3.11. *Any fuzzy dot ideal μ of BCK-algebra X with $\mu(0) = 1$ must be a fuzzy dot subalgebra of X .*

Proof. Since $xy \leq x$, then by Proposition 3.7., $\mu(x) \leq \mu(xy)$. Thus $\mu(xy) \geq \mu(x) > \mu(x)\mu(y)$.

Theorem 3.12. *Let $\{\mu_i\}$, where $i \in \mathrm{I}$ be a family of fuzzy dot ideals of a BCK/BCI-algebra X , then so is $\bigcap_{i \in I} \mu_i$.*

Proof. For all $x, y \in X$, we get

$$\begin{aligned} \bigcap_{i \in I} \mu_i(0) &= \min_{i \in I}\{\mu_i(0)\} \\ &\geq \min_{i \in I}\{\mu_i(x)\} \\ &= \bigcap_{i \in I} \mu_i(x) \end{aligned}$$

$$\begin{aligned}\underset{i\in I}{\cap}\mu_i(x) &= \min_{i\in I}\{\mu_i(x)\} \\ &\geq \min_{i\in I}\{\mu_i(xy)\mu_i(y)\} \\ &\geq \left(\min_{i\in I}\{\mu_i(xy)\}\right)\left(\min_{i\in I}\{\mu_i(y)\}\right) \\ &= \left(\underset{i\in I}{\cap}\mu_i(xy)\right)\left(\underset{i\in I}{\cap}\mu_i(y)\right)\end{aligned}$$

Hence $\underset{i\in I}{\cap}\mu$ is a fuzzy dot ideal of X .

Remark 3.13. Note that a fuzzy subset μ of a *BCK/BCI*-algebra X is a fuzzy ideal of X if and only if a nonempty level subset μ_t is an ideal of X for every $t\in[0,1]$. But if μ is a fuzzy dot ideal of X , then μ_t may not to be an ideal of X , as seen in the following example.

Example 3.14. Let $X=\{0,a,b,c\}$ be a *BCK*-algebra as defined in Example 3.4. Consider the same fuzzy dot ideal ν of X which is defined by $\nu(0)=\nu(a)=0.5$, $\nu(b)=0.4$ and $\nu(c)=0.3$. We can see that $\nu_{0.5}=\{0,a\}$ and $ba=a\in\nu_{0.5}$, but $b\notin\nu_{0.5}$, then $\nu_{0.5}$ is not an ideal of X .

Theorem 3.15. *Let μ be a fuzzy dot ideal of BCK/BCI-algebra X . Then $X_\mu=\{x\in X\mid\mu(x)=1\}$ is either empty or ideal of X .*

Proof. Suppose that μ is a fuzzy dot ideal of X , clearly $0\in X_\mu$, now let $X_\mu\neq\phi$, and xy , $y\in X_\mu$. Then $\mu(xy)=1=\mu(y)$, so $\mu(x)\geq\mu(xy)\mu(y)=1$ gives $x\in X_\mu$. Hence X_μ is an ideal of X .

Theorem 3.16. *Let $g:X\to X'$ be a homomorphism of BCK/BCI-algebras. If ν is a fuzzy dot ideal of X', then the preimage $g^{-1}(\nu)$* of *ν under g is a fuzzy dot ideal of X .*

Proof. For any $x,y\in X$, we have

$$g^{-1}(\nu)(0)=\nu(g(0))\geq\nu(g(x))=g^{-1}(\nu)(x)$$

$$\begin{aligned}g^{-1}(\nu)(x) &= \nu(g(x)) \\ &\geq \nu(g(x)(g(y)))\nu(g(y)) \\ &= \nu(g(xy))\nu(g(y)) \\ &= g^{-1}(\nu(xy))g^{-1}(\nu(y))\end{aligned}$$

Hence $g^{-1}(\nu)$ is a fuzzy dot ideal of X .

Theorem 3.17. *For any fuzzy subset σ of BCK/BCI-algebra X , assume that μ_σ be a fuzzy subset of $X\times X$ defined by $\mu_\sigma(x,y)=\sigma(x)\sigma(y)$ for all $x,y\in X$. Then σ is a fuzzy dot ideal of X if and only if μ_σ is a fuzzy dot ideal of $X\times X$.*

Proof. Assume that σ is a fuzzy dot ideal *of* X .For all $x\in X$, we have

$$\mu_\sigma(0,0)=\sigma(0)\sigma(0)\geq\sigma(x)\sigma(x)=\mu_\sigma(x,x)\cdot$$

For any $x_1,x_2,y_1,y_2\in X$, we have

$$\begin{aligned}
&\mu_\sigma\left((x_1,x_2)(y_1,y_2)\right)\mu_\sigma(y_1,y_2)\\
&=\mu_\sigma(x_1y_1,x_2y_2)\mu_\sigma(y_1,y_2)\\
&=\left(\sigma(x_1y_1)\sigma(x_2y_2)\right)\left(\sigma(y_1)\sigma(y_2)\right)\\
&=\left(\sigma(x_1y_1)\sigma(y_1)\right)\left(\sigma(x_2y_2)\sigma(y_2)\right)\\
&\le\sigma(x_1)\sigma(x_2)\\
&=\mu_\sigma(x_1,x_2),
\end{aligned}$$

And so μ_σ is a fuzzy dot ideal of $X\times X$.

Conversely, suppose that μ_σ is a fuzzy dot ideal of $X\times X$ and let $x,y\in X$. Then

$$\begin{aligned}
\left(\sigma(xy)\sigma(y)\right)^2&=\left(\sigma(xy)\sigma(y)\right)\left(\sigma(xy)\sigma(y)\right)\\
&=\left(\sigma(xy)\sigma(xy)\right)\left(\sigma(y)\sigma(y)\right)\\
&=\mu_\sigma(xy,xy)\mu_\sigma(y,y)\\
&=\left(\mu_\sigma(x,x)\mu_\sigma(y,y)\right)\mu_\sigma(y,y)\\
&\le\mu_\sigma(x,x)\\
&=\sigma(x)\sigma(x)=\left(\sigma(x)\right)^2
\end{aligned}$$

And so $\sigma(x)\ge\sigma(xy)\sigma(y)$, that is σ a fuzzy dot ideal of X .

Theorem 3.18. *Let X be a BCK/BCI-algebra, and let μ be a fuzzy set of $X\times X$ and σ be a fuzzy subset of X defined by $\sigma(x)=\mu(x,0)$, for all $x\in X$. If μ* is a *fuzzy dot ideal of $X\times X$, then σ* is a *fuzzy dot ideal of X* .

Proof. For all $x\in X$ we have

$\sigma(0)=\mu(0,0)\ge\mu(x,0)=\sigma(x)$. For all $x,y\in X$

$$\begin{aligned}
\sigma(xy)\sigma(y)&=\mu(xy,0)\mu(y,0)\\
&=\mu(xy,00)\mu(y,0)\\
&=\mu\left((x,0)(y,0)\right)\mu(y,0)\\
&\le\mu(x,0)\\
&=\sigma(x)
\end{aligned}$$

Thus σ is a fuzzy dot ideal of X .

Theorem 3.19. *Let X, X' be BCK/BCI-algebras, and μ a fuzzy set of $X\times X'$ satisfying the inequalities $\mu(x,0)\ge\mu(x,x')$ and $\mu\left((x,0)(y,0)\right)\ge\mu\left((x,x')(y,y')\right)$ for all $x,y\in X$ and $x',y'\in X'$. Let σ be a fuzzy subset of X defined as above. If σ* is a *fuzzy dot ideal of X* , then μ is a *fuzzy dot ideal of $X\times X'$*.

Proof. For all $(x,y)\in X\times X'$, we have

$$\mu(0,0)=\sigma(0)\ge\sigma(x)=\mu(x,0)\ge\mu(x,y),$$

and for all $(x,x'),(y,y')\in X\times X'$

$$\begin{aligned}\mu(x,x') &= \sigma(x) \geq \sigma(xy)\sigma(y) \\ &= \mu(xy,0)\mu(y,0) \\ &= \mu(xy,00)\mu(y,0) \\ &= \mu((x,0)(y,0))\mu(y,0) \\ &\geq \mu((x,x')(y,y'))\mu(y,y')\end{aligned}$$

Thus μ is a fuzzy dot ideal of $X \times X'$.

Theorem 3.20. *Let* μ *and* ν *be fuzzy dot ideals of a BCK/BCI-algebras* X *and* X' *respectively. Then the cross product* $\mu \times \nu$ *of* μ *and* ν *defined by* $\mu \times \nu(x,y) = \mu(x)\nu(y)$, *for all* $(x,y) \in X \times X'$ *is a fuzzy dot ideal of* $X \times X'$.

Proof. For all $(x,y) \in X \times X'$ we have

$$\mu \times \nu(0,0) = \mu(0)\nu(0) \geq \mu(x)\nu(y) = \mu \times \nu(x,y)$$

Now, for any $(x,x'),(y,y') \in X \times X'$, we have

$$\begin{aligned}\mu \times \nu(x,x') &= \mu(x)\nu(x') \\ &\geq (\mu(xy)\mu(y))(\nu(x'y')\nu(y')) \\ &= (\mu(xy)\nu(x'y'))(\mu(y)\nu(y')) \\ &= (\mu \times \nu(xy,x'y'))(\mu \times \nu(y,y')) \\ &= (\mu \times \nu(x,x')(y,y'))(\mu \times \nu(y,y'))\end{aligned}$$

Thus $\mu \times \nu$ is a fuzzy dot ideal of $X \times X'$.

Theorem 3.21. *Let* μ *and* ν *be fuzzy dot ideals of BCK/BCI-algebras* X *and* X' *respectively. If the cross product* $\mu \times \nu$ *is a fuzzy dot ideal of* $X \times X'$, *then* μ *or* ν *must be a fuzzy dot ideal .*

Proof. Let $\mu \times \nu$ *be a fuzzy dot ideal of* $X \times X'$. We claim that μ or ν satisfies (FD-1*).*Suppose $\mu(0) < \mu(x_0)$ and $\nu(0) < \nu(x'_0)$, for some $x_0 \in X$ and $x'_0 \in X'$.Then

$$\mu \times \nu(0,0) = \mu(0)\nu(0) < \mu(x_0)\nu(x'_0) = \mu \times \nu(x_0,x'_0)$$

which is a contradiction. Therefore (FD-1) holds for one μ or ν . Suppose that (FD-2) is false. Then there are $x_0, y_0 \in X$ and $x'_0, y'_0 \in X'$ such that

$$\begin{aligned}\mu \times \nu(x_0,x'_0) &= \mu(x_0)\nu(x'_0) \\ &< (\mu(x_0y_0)\mu(y_0))(\nu(x'_0y'_0)\nu(y'_0)) \\ &= (\mu(x_0y_0)\nu(x'_0y'_0))(\mu(y_0)\nu(y'_0)) \\ &= \mu \times \nu(x_0y_0,x'_0y'_0)\mu \times \nu(y_0,y'_0) \\ &= \mu \times \nu((x_0,x'_0)(y_0,y'_0))\mu \times \nu(y_0,y'_0)\end{aligned}$$

Which is impossible. Hence (FD-2) is also valid for one μ or ν. Consequently, μ or ν must be a fuzzy dot ideal.

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