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Research Article

SOME APPLICATIONS OF THE NON - POSITIVE SECTIONAL CURVATURE

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Abstract

The Applications of Non –positive Sectional Curvature plays a great role in the field of Physics, Mathematics and engineering because it paves to the knowledge of radius of curvature, length of curvature, and arc of curvature. The study aims to explain some applications of non- positive sectional curvature. We followed the analytical induction mathematical method. We found the following some results: The sectional curvature indicates to know the behavior of some of the functions and the solution of mathematics equations, and also it reveals the Euler – Lagrange equation which is considered one of the importance application of the non - positive sectional curvature.

Keywords: Non –positive Sectional Curvature.

INTRODUCTION

We know that any two points in a connected, simply connected, complete manifold M of constant negative curvature can be connected by a unique geodesic. Thus, the entire manifold M is geodetically convex and its infectivity radius is infinity. This continues to hold in much greater generality for manifolds with non positive sectional curvature. It is convenient, at this point, to extend the discussion to Riemannian manifolds in the intrinsic setting. In particular, at some point in the proof of the main theorem of this section and in our main example, we shall work with a Riemannian metric that does not arise (in any obvious way) from an embedding. Euler-Lagrange equations.

2. Differentiable Spaces with Non-Positive Curvature:

The connection of present definition of non-positive curvature with the standard definition can easily be discussed by using the following lemma

Lemma (2.1): If in a space with non-positive curvature, x(t) and y(t) represent geodesics with x(0) = y(0) then

$$x(\alpha t)y(\beta t)t = \mu(\alpha , \beta) \qquad (1)$$
$$\alpha, \beta \neq 0$$

Exists $\mu(\alpha, \beta) \leq \lfloor\alpha\rfloor + \lfloor\beta\rfloor$
And x$(\alpha t) y(\beta t) \geq t\mu(\alpha\ \beta)$ for small positive t.

For x$(\alpha t)\ y(\beta t)$ is a convex function of t and has therefore at t=0 a right hand derivative $\mu(\alpha , \beta)$. The relation $\mu(\alpha , \beta) \leq \lfloor\alpha\rfloor + \lfloor\beta\rfloor$ follows from x$(\alpha t)y(\beta t) \leq for$ t$> o$, and follows form the fact ,that a convex function lies a above the right hand tangent at any of its point. In Riemann spaces non-positive curvature is equivalent to the ≤ cosine inequality≥ which can be formulated under very weak differentiability hypotheses

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Lemma (2.2): A Riemann space has non-positive curvature in the present sense if and only if it has non-positive curvature in the usual sense.

Proof

If a Riemann space has non-positive curvature in the usual sense ,then holds locally ,where it is proved that holds in the large for simply connected spaces. This implies ,of course, that it hold in the small for general spaces. By the space has non-positive curvature in the present sense. The converse can be proved to establish and then tracing Cartan s steps back. But it is simpler and geometrically more convincing the proceed as follows let x(t) and y(t) represent two geodesics which form at x(0) = y(0) the angle γ. Then as in preceding proof

$$\frac{x(t)y(t)}{t} = \mu\ (1\ ,1) = [\![2(1 - cos\gamma)]\!]^{\frac{1}{2}} = 2\sin\left(\frac{\gamma}{2}\right) \qquad (2)$$

Hence by

X(t) y(t) ≥ 2t sin $(\frac{\gamma}{2})$.

That R has at p non-positive curvature means this if p is any two dimensional surface element at p then the two dimensional surface μ foemed by all geodesics through p and tangent to p has at p a non-positive Gauss curvature.

Take γ geodesics x_v(t) in μ through p such that the angle formed by x_v(t) and x_v(t)at p is $\frac{2\pi}{v}$. If λ is the length of circle with radius t about p in μ then

$\lambda > \sum \qquad x_v(t)x_{v+1}(t) \geq 2vt\ sin\ (\frac{\pi}{v})$, hence

$\lambda \geq 2\pi t\ for\ v \to\ \infty$. the well known expression of Bertrand and Puiseux for the Gauss curvature k

$$K = 3\pi^{-1}(2\pi t - \lambda)t^{-3} \qquad (3)$$

Shows that $K \leq 0$.

3. Manifolds of Constant Curvature:

Definition (3.1): By definition, an n –dimensional manifold of constant curvature κ is a length space X that is locally isometric to M_k^n. In other words, for every point x ∈ X there is an ε> 0 and an isometry φ from B (x ,ε) onto a ball B (φ (x) ,ε) ⊂ M_k^n.

Theorem (3.2): Let X be a complete, connected, n-dimensional manifold of constant curvature κ. When endowed with the induced length metric, the universal covering of X is isometric to M_k^n.

Proof

The following proof is due to C. E hresman. In the first part of the proof we do not assume that X is complete. By definition, a chart φ : u → M_k^n is an isometry from an open set U ⊆ X onto an open set φ(u) ⊆ M nκ. If φ : u → M nκ is another chart and if u ∩ $\underline{u}$ is connected, then there is a unique isometry g ∈ Isom (M nκ) such that φ and g ◦ φ are equal on u ∩ $\underline{u}$. Consider the set of all pairs (φ, x), where φ : u → M nκ is a chart and x ∈ u. We say that two such pairs (φ, x) and (φ, x) are equivalent if x = $\dot{x}$ and if the restrictions of φ and $\acute{\varphi}$ to a small neighbourhood of x coincide. This is indeed an equivalence relation and the equivalence class of (φ, x) is called the germ of φ at X. Let ˆ X be the set of all equivalence classes, I e. the set.[3] pp45to46

4. Euler-Lagrangian equations:

A curve $z(\phi)$ which minimises L (or, more generally, renders it stationary with respect to small variations)satisfies the Euler-Lagrange equation

$$\frac{d}{d\phi}\left(\frac{\partial f}{\partial z'}\right) = \frac{\partial f}{\partial z} \qquad (4.1)$$

where $f = ((1+k^2)z^{r2} + k^2z^2)^{1/2}$. It is convenient to divide f by the constant$(1+k^2)^{1/2}$. Note that thisleaves the Euler-Lagrange equation unchanged, and amounts to rescaling the length L by a ¯x end factor. Let

$$a^2 = \frac{k^2}{1+k^2} = \frac{tan^2\alpha}{1+tan^2\alpha} = \frac{tan^2\alpha}{sec^2\alpha} = sin^2\alpha \qquad (4)$$

So that

$$a = sin\, sin\, \alpha \qquad (5)$$

Then we may take

$$f = (z^{r2} + a^2z^2)^{1/2} \qquad (7)$$

Since f does not depend explicitly on ϕ, we may use the alternative form of the Euler-Lagrange equation,

$$f - \frac{\partial f}{\partial z'}z' = c, const \qquad (8)$$

After some calculation, this gives

$$\frac{a^2z^2}{(z^{r2} + a^2z^2)^{1/2}} = c \qquad (9)$$

To solve the Euler-Lagrange equation, first observe that a specific case is $z_1 = z_2$, i.e. the endpoints ofthe curve are on the same height. Then (9) admit s a trivial solution $z(\phi) = const$. However, it is in fact not a solution of the Euler-Lagrange equation – this can be verified directly – unless c = 0, hence z = 0! Indeed, consider the alternative form of the Lagrange equation (14) and differentiate it with respect to time. The result will be: either z'= 0, or $\frac{d}{dt}f_{z'} = f_z$, the Euler-Lagrange equation. Thus, the alternative form of the Lagrange equations can generate a "rubbish" solution z = const; which should always be checked with the Euler-Lagrange equation $\frac{d}{dt}f_{z'} = f_z$. It is easy to see that if f is given by (7), the only constant solution of the Euler-Lagrange equation is z = 0, corresponding to the apex of the cone.

But there are other, less trivial solutions of (9), and they are minimizers. This will not be show explicitly, but can be done using the unwrapping argument – after the cone having been unwrapped, the solutions we are about to find will become straight line segments that shall be glued along the edges of the sector, in the way described in the preamble. Let us express z0 in terms of z from (9):

$$\frac{dz}{d\phi} = \pm az\sqrt{\frac{a^2}{c^2}z^2 - 1}$$

In the boundary conditions (3.19), without loss of generality, we can assume $z_1 \geq z_2$. Then "+" corresponds to curves which go up (z increasing) in the anticlockwise (increasing- ∅) direction, and a "–" to curves which go up in the clockwise direction. It suffices to consider the "+" case, and we'll make the necessary remarks about the "–" case along the way after we see what's going on. In (3.20) variables separate, and it can be integrated as follows (using indefinite integrals for convenience)

$$\int \frac{dx}{z\sqrt{a^2z^2/c^2 - 1}} = a\phi + z_1 \geq u_1$$

where $u_1$is some constant of integration. Recall that

$$\int \frac{dz}{z(z^2-1)^{1/2}} = sec^{-1}z,$$

the inverse secant. Indeed, a trigonometric substitution $z = sec\, u$yields $dz = \frac{sinsin\, udu}{cos^2\, u}$ and reduces the aboveintegral to simply$\int \quad du$. Similarly,

$$\int \frac{dx}{z\sqrt{a^2z^2/c^2 - 1}} = sec^{-1}[(a/c)z].$$

Thus

$$z\phi = \frac{c}{a\, cos\, cos\, (a\phi + u_1)}$$

C and $u_1$are constants to be determined from the boundary conditions. The constant $u_1$is clearly definedup to a multiple of 2π, and in fact one can assume $|u| < \frac{\pi}{2}$. Adding $\pi$to $u_1$would negate the cosine and correspond to the choice of the "–" sign in (3.20). Also note that $u_1$maybe not allowed to change continuously over an interval of values of length π and longer, because otherwise, given ϕ, the denominatorof (3.22) will zero at some point: e.g. the secant is only defined for $|a\phi + u_1| < \pi = 2$.

5. Euler Equation and Geodesics

5.1. Variational Problems:

The term calculus of variations was first coined by Euler in 1756 as a description of the method that Joseph Louis Lagrange had introduced the previous year. The method was since expanded and studied by Euler, Hamilton, and others. As noted at the beginning of the chapter, the main idea is to determine which functions y(x) will minimize, or maximize, integrals of the form

$$J[y] = \int_b^a f\left(x, y(x), y'(x)\right)dx,$$

$$\left.\frac{d\phi}{d\epsilon}\right|_{\epsilon=0} = 0.$$

Where a, b, f (a), f (b), and f (x) are given. Integrals like J[y] are called functional s. This is a mapping from a function space to a scalar. J takes a function and spits out a number. An example of a functional is the length of a curve in the plane. As we will recall, the length of a curve y = y(x) from x = x1 to x = x2 is given by

$$L[y] = \int_{x1}^{x2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

This maps a given curve y(x) to a number, L.

We are interested in finding the extrema of such functional s. We will not formally determine if the extrema are minimal or maxima. However, in most cases, it is clear which type of extrema comes out of the analysis. Further analysis of the second variation can be found elsewhere, such as Lancz s '. For interesting problems, historical principle shave led to the formulation of problems in the calculus of variations, such as the Principles of Least Time, Least Action, Least Effort, or the shortest dis Euler equation and geodesics 3tance between geometric points on a surface. We will explore some standard examples leading to finding the extrema of functional s.

6. Euler Equations

In the previous examples we have reduced the problems to finding functions $y = y(x)$that extremize functionals of the form

$$J[y] = \int_{x1}^{x2} |r'(x)|dx.$$

for F twice continuous (c2) in all variables. Formally, we say that J is stationary,

$$\delta J[y] = 0$$

at the function $y = y(x)$, or that the function $y = y(x)$that extremizes$J[y]$satisfies

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

This is called the Euler's Equation. We will derive Euler's Equation and then show how it is used for some common examples. The idea is to consider all paths connected to the two fixed points and finding the path that is an extremism of $J[y]$. In fact, we need only consider parametizing paths near the optimal path and writing the problem in a form that we can use with the methods for local extrema of real functions.

Let's consider the paths $y(x;\epsilon) = u(x) + \epsilon y(x)$near the optimal path$u = u(x)$with $y(a) = y(b) = 0$and $y$is C2. Then, we consider the functional

$$J[y] = J[u + \epsilon y] \equiv \emptyset(\epsilon)$$

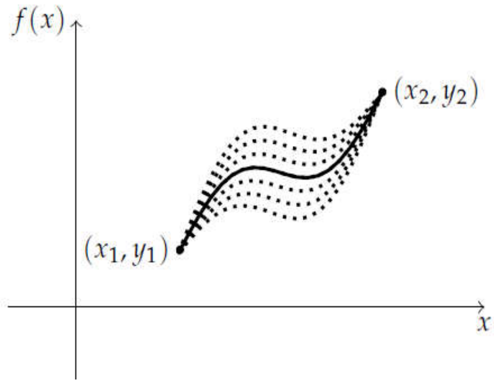

Figure 1. Paths near an optimal path between two fixed points

We note that if $J[u + \epsilon y]$has a local extremum at u, then u is a stationary function for J. This will occur when

We compute this derivative and find

$$\frac{d\emptyset}{d\epsilon} = \frac{d}{d\epsilon}\int_a^b F(x, u + \epsilon\eta, u' + \epsilon\eta')dx$$

$$= \int_a^b \frac{\partial}{\partial\epsilon}F(x, u + \epsilon\eta, u' + \epsilon\eta')dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial y}\frac{\partial y}{\partial\epsilon} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial\epsilon}\right]dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial\epsilon}\eta'(x)\right]dx$$

We can perform an integration by parts on the second integral in order to move the derivative off of$\eta'(x)$. This is accomplished by setting $u(x) = \frac{\partial F}{\partial y'}$and $dv = \eta'(x)dx$ in the integration by parts formula. Then,

$$\int_a^b \frac{\partial F}{\partial y'}\eta'(x)dx = \eta(x)\frac{\partial F}{\partial y'}| - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)dx$$

The first terms vanish because$\eta(a) = \eta(b) = 0$.

This leaves

$$\frac{d\emptyset}{d\epsilon} = \int_a^b \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right]dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta'(x)dx$$

Evaluation at $\epsilon = 0$ gives

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta'(x)dx = 0.$$

Noting that $\eta(x)$is an arbitrary function and that this integral vanishes for all $\eta(x)$, we can say that the integrand vanishes,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0,$$

for all $x \in [a, b]$. This is Euler's Equation, (3).

Because $F = F(x, y(x), y'(x))$, one can prove a second form of Euler's

Equation. We first note for the Chain Rule that

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x}$$

Now we insert

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

from Euler's Equation to find

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x}$$

$$= \frac{\partial F}{\partial x} + \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)y' + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x}$$

Rearranging this result, we obtain

$$\frac{\partial F}{\partial x} - \frac{d}{dx}\left(F - \frac{\partial F}{\partial y'}y'\right) = 0$$

This is the second form of Euler's Equation.

The second form of Euler's Equation is handy when $\frac{\partial F}{\partial x} = 0$,or when

$F = F(y, y')$is independent of x. In this case, we have

$$F - \frac{\partial F}{\partial y'}y' = c,$$

Where c is an arbitrary constant.

Special Cases:

$$F(x, y) \Rightarrow \frac{\partial F}{\partial y} = 0$$

$$F(x, y') \Rightarrow \frac{\partial F}{\partial y'} = c$$

$$F(x, y') \Rightarrow F - \frac{\partial F}{\partial y'}y' = c$$

$$F(y') \Rightarrow y'' = 0$$

There are other special cases. In Euler's Equation if $F = F(x, y')$,

$$\frac{\partial F}{\partial y'} = c$$

and when $F = F(x, y)$, we have

$$\frac{\partial F}{\partial y} = 0$$

Finally, when $F = F(y')$, Euler's Second Equation implies

$$y'' = 0$$

or $y(x) = c_1 + c_2 x$.

Example (6.1):

Determine the closed curve with a given fixed length that encloses the largest possible area. The area enclosed by curve c is given by

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)dt.$$

This was verified as an example of Green's Theorem in the Plane, as

$$\oint_C x\,dy - ydx = \int_S \left(\frac{\partial F}{\partial y} - \frac{\partial(-y)}{\partial y}\right) - dxdy = 2\int_S dxdy$$

If the curve is parameterized as $x = x(t)$and $y = y(t)$, then

$$A = \frac{1}{2}\oint_C (x\dot{y} - y\dot{x})dt$$

Also, the length of the curve is given as

$$L = \frac{1}{2}\oint_C \sqrt{(\dot{x}^2 - \dot{y}^2)}dt$$

The problem can be mathematically stated as finding the maximum of $A[x, y]$subject to the constraint $L[x, y]$. This is equivalent to finding the path for which $I[x, y] = A[x, y] + \lambda L[x, y]$is stationary for some constant λ. Thus, we consider the integral

$$I[x, y] = \oint_C \left(\frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{(\dot{x}^2 - \dot{y}^2)}\right)dt.$$

We define the integrand

$$F(t, x, \dot{x}, y, \dot{y}) = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{(\dot{x}^2 - \dot{y}^2)}$$

The variational problem states that F satisfies a pair of Euler-Lagrange equations:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) + 0,$$

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) + 0$$

Inserting F, we have

$$\frac{1}{2}\dot{y} - \frac{d}{dt}\left(\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0,$$

$$-\frac{1}{2}\dot{x} - \frac{d}{dt}\left(\frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0$$

Each of these equations is a perfect derivative and can be integrated.

The results are

$$y - \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = c1,$$

$$x - \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = c2,$$

Where c1 and c2 are two arbitrary constants. Rearranging, squaring both equations, and adding, we ha

$$(x-c2)^2 + (y-c1)^2 = \frac{\lambda^2\dot{x}^2}{\dot{x}^2+\dot{y}^2} + \frac{\lambda^2\dot{y}^2}{\dot{x}^2+\dot{y}^2} = \lambda^2$$

Therefore, the maximum area is enclosed by a circle [2].

7. Geodesics

The shortest distance between two points is a straight line. Well, this is true in a Euclidean plane. But, what is the shortest distance between two points on a sphere? What path should light follow as it passes the sun?

Paths that are the shortest distances between points on a curved surface or events in curved space time are called geodesics. In general, we can set up an appropriate integral to compute these distances along paths and seek the paths that render the integral stationary. We begin by looking at the geodesics on a sphere.

Example (7. 1): Find the Christoffel symbols for the surface of a sphere.

This is an example of how the general geodesic computation can be used for Riemannian metrics. First, we look at the geodesics found in

$$sin\,sin\,\theta\;\;cos\,cos\,\theta\left(\frac{d\phi}{ds}\right)^2 - \frac{d^2\theta}{ds^2} = 0$$

$$\frac{d}{ds}\left(\theta\,\frac{d\phi}{ds}\right) = 0.\;(3.66)$$

Solving for the second-order derivatives, we have

$$\frac{d^2\theta}{ds^2} - sin\,sin\,\theta\;\;cos\,cos\,\theta\left(\frac{d\phi}{ds}\right)^2 = 0,$$

$$\frac{d^2\phi}{ds^2} + \theta\,\frac{d\theta}{ds}\frac{d\phi}{ds} = 0$$

The expanded forms of the geodesic equations for $\theta$and $\phi$are

$$\frac{d^2\theta}{ds^2} + T^{\theta}_{\theta\theta}\frac{d\theta}{ds}\frac{d\theta}{ds} + T^{\theta}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} + T^{\theta}_{\phi\theta}\frac{d\phi}{ds}\frac{d\theta}{ds} + T^{\theta}_{\phi\phi}\frac{d\phi}{ds}\frac{d\phi}{ds} = 0$$

$$\frac{d^2\phi}{ds^2} + T^{\phi}_{\theta\theta}\frac{d\theta}{ds}\frac{d\theta}{ds} + T^{\phi}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} + T^{\phi}_{\phi\theta}\frac{d\phi}{ds}\frac{d\theta}{ds} + T^{\phi}_{\phi\phi}\frac{d\phi}{ds}\frac{d\phi}{ds} = 0$$

Comparing these with the geodesics, we have that

$T^{\theta}_{\phi\phi} = -\,sin\,sin\,\theta\;\;cos\,cos\,\theta, T^{\phi}_{\theta\phi} = T^{\phi}_{\phi\theta}$ $=cot\,cot\,\theta$,

and the rest of the Christoffel symbols vanish.

Now we use Equation (34) to compute the Christoffel symbols. We first note that the metric is given by

$$g = (1\;0\;0\;\theta\;)$$

Because g is diagonal, the coefficients are relatively easy to find. In$g_{\alpha\gamma}T^{\alpha}_{\delta\beta}$, we note that $\alpha = \gamma$for non-vanishing contributions from the metric. So, this gives for $\gamma = 0$

$$g_{\theta\theta}T^{\theta}_{\delta\beta} = \frac{1}{2}\left[\frac{dg_{\delta\theta}}{dx^{\beta}} + \frac{dg_{\theta\beta}}{dx^{\delta}} - \frac{dg_{\delta\beta}}{dx^{\theta}}\right]$$

Because g is independent of ϕ, all ϕderivatives will vanish. Also, $g_{\phi\phi}$is the only coefficient that depends on θ. So, the only time the right side of the equation does not vanish is for $\delta = \beta = \phi$. This leaves

$$T^{\phi}_{\phi\theta} = -\frac{dg_{\phi\phi}}{dx^{\theta}} = -\,sin\,sin\,\theta\;\;cos\,cos\,\theta$$

Similarly, for $\gamma = \phi$we have

$$g_{\phi\phi}T^{\phi}_{\phi\theta} = \frac{1}{2}\left[\frac{dg_{\delta\phi}}{dx^{\beta}} + \frac{dg_{\phi\beta}}{dx^{\delta}} - \frac{dg_{\delta\beta}}{dx^{\phi}}\right],$$

Using the same arguments about the derivative of the metric elements, we find one of βor $\delta$is $\phi$and the other is θ. For example,

$$g_{\phi\phi}T^{\phi}_{\phi\theta} = \frac{1}{2}\left[\frac{dg_{\phi\phi}}{dx^{\theta}} + \frac{dg_{\phi\theta}}{dx^{\phi}} - \frac{dg_{\phi\theta}}{dx^{\phi}}\right],$$

$$\theta\,T^{\phi}_{\phi\theta} = \frac{1}{2}(2\,sin\,sin\,\theta\;\;cos\,cos\,\theta\;),$$

$$T^{\phi}_{\phi\theta} = cot\,cot\,\theta$$

Because $T^{\phi}_{\phi\theta} = T^{\phi}_{\theta\phi}$,we have obtained the same results based on reading the geodesic equation [4].

8. The Levi-Civita Connection and its curvature

The Einstein summation convention and the Ricci Calculus When dealing with tensors on a manifold it is convenient to use the following conventions. When we choose a local frame for the tangent bundle we writee1,... en for this basis. We always index bases of the tangent bundle with indices down. We write then a typical tangent vector

$$X = \sum_{i=1}^{n} \;X^i e_i.$$

Einstein's convention says that when we see indices both up and down we assume that we are summing over them so he would write

$X = X^i e_i$

While a one form would be written as

$$\theta = a_i e^i$$

Where e i is the dual co-frame field. For example when we have coordinates

x1, x2,…, x n then we get a basis for the tangent bundle $\partial/\partial x^i, \ldots, \partial/\partial x^n$

More generally a typical tensor would be written as

$$T = T^i_{jk} l e_i e^j e^k \; e_l$$

Note that in general unless the tensor has some extra symmetries the order of the indices matters. The lower indices indicate that under a change of frame $f_i = C_i{}^j e_j$ a lower index changes the same way and is called covariant while an upper index changes by the inverse matrix. For example the dual co-frame field to the f I, called f I is given by

$$f_i = D_j{}^i e^j$$

Where $D_j{}^i$ is the inverse matrix to $C_i{}^j$ (so that $D_j{}^i C_k{}^j = \delta_k{}^i$.) The components of the tensor T above in the f i basis are thus

$$T^i_{jk} l = T^{i'}_{j'k'} l' D^i_{i'} C^{j'}_j C^{k'}_k D^l_{l'}$$

Notice that of course summing over a repeated upper and lower index results in a quantity that is independent of any choices. Given a vector bundle over our manifold which is not the tangent bundle or tensors on the tangent bundle we use a distinct set of indices to indicate tensors with values on that bundle. If V →M is a vector bundle of rank k with a local frame v a, 1 ≤ α ≤ r we would write

$$s = c^a_i v_a \epsilon dx^i$$

for a typical section of the bundle T* M ⋴ V

Given a ∇ connection in V we write

$$\nabla s = s^a_i \; dx^i \epsilon v^\alpha$$

That is we think ∇s as a section of $T * M \epsilon V$as opposed to the possiblymore natural $V \epsilon \, T * M$. Or more concretely the semi-colon is also indicating that the indices following the semi-colon are to really be thought of coming first and the opposite order. Our convention here is designed to be more consistent with the mathematical literature. In the physics literature for example "Graviation" by Misner, Throne and Wheeler. So for a connection in the tangent bundle if we have a vector field with components X_i we have write X^i_j for the components of its covariant derivative. The Christoffelsymbols of a connection are the components of the covariant derivatives of the basis vectors:

$$\nabla_{ei} v^\alpha = T^a_i b v^\beta$$

Then we can write more explicitly

$$s^a_i = e_i s^\alpha + \; T^a_i \beta s^\beta$$

One often uses the short hand

$$e_i s^\alpha = s^a_i$$

so that

$s^a_i = s^a_i + T^a_i \beta s^\beta$ [1]

RESULTS

The sectional curvature indicates to know the behavior of some of the functions and the solution of mathematics equations, and also it revealed the Euler – Lagrange equation which is considered one of the application of the non - positive sectional curvature.

REFERENCES

1. Department of Mathematics Geometry of Manifolds, 18.966 Spring 2005 Lecture Notes.
2. John Operaa Differential Geometry and its Applications Cleveland State University 2007.
3. Martin R Bridson, Matric Space of Non –Positive Curvature, Geneva March, 1999.
4. R Heman Euler Equation and Geodesics, February 2018
5. Benakli, N. Polygonal complexes I: combinatorial and geometric properties, Preprint, Princeton University (1993).
6. Berestovskiˇı N. & I.G. Nikolaev, Multidimensional generalized Riemannian spaces, Geometry IV (Yu.G. Reshetnyak, ed.), *Encyclopaedia of Math. Sciences* 70, Springer-Verlag, Berlin-Heidelberg-New York..., 1993, pp. 165–243.
7. Hitchin, N. A new family of Einstein metrics, Manifolds and geometry (Pisa, 1993), 190-222,
8. Korevaar N. & R. Schoen, Sobolev spaces and harmonic maps for metric space targets,Communications in Analysis and Geometry 1- (1993), 561–659.
9. Mok N., Siu YT. & S- K. Yeung, Geometric superrigidity Inventionesmath. 113 –(1993), 57- 83.
10. Ziller, W. Fatness revisited, lecture Notes 2000, unpublished University of Pennsylvania Philadelphia, PA 19104.
11. Ziller, W. Homogeneous spaces, biquotients, and manifolds with positive curvature, Lecture Notes1998, unpublished.
12. Walschap G. Metric Structures in Differential Geometry. Series: Graduate Texts in Mathematics, Vol. 224. Bir*ch* auser ,2004.
13. Willmore T.J. Riemannian Geometry. Oxford University Press, 1997.
14. Bazaikin, Y. A manifold with positive sectional curvature and fundamental group Z3©Z3, *Siberian Math. J.* 40 - (1999), 834-{836.
15. Bazaikin, Y. On a family of 13-dimensional closed Riemannian manifolds of positive curvature, *Siberian Math. J., 37-* (1996), 1068-1085.

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