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**Research Article**

# APPLICATION ON LYAPUNOV-TYPE INEQUALITIES FOR PLANAR LINEAR DYNAMIC HAMILTONIAN SYSTEMS

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**Abstract**

We present an application on a new Lyapunov-type inequalities given by Martin Bohner and Agacık Zafer [13] for linear Hamiltonian systems onarbitrary time scales, which improve many results and the related ones in the literature. As an application, we obtain new disconjugacy criteria for linear Hamiltonian systems.

**Keywords:** Agriculture, Development, Issues and Challenges, Strategy, Mutation.

## INTRODUCTION

We show an application on the establish Lyapunov-type inequalities for the planar Hamiltonian system established by Martin Bohner and Agacık Zafer [13]

$$x^{\Delta}=\sum_{j}\alpha_j(a+\epsilon)x^{\sigma}+\sum_{j}\beta_j(a+\epsilon)u,\qquad u^{\Delta}=-\sum_{j}\gamma_j(a+\epsilon)x^{\sigma}-\sum_{j}\alpha_j(a+\epsilon)u, \quad (1.1)$$

where $\alpha_j,\beta_j,\gamma_j$ are sequences of real-valued rd-continuous functions defined on a given arbitrary time scale $\mathbb{T}$.

Lyapunov-type inequalities have proved to be very useful in studying the qualitative behavior of solutions such as oscillation, disconjugacy, and eigenvalue problems for differential and difference equations. Although Lyapunov-type inequalities are well developed for the continuous case after the appearance of Lyapunov's well-known inequality, discrete Lyapunov-type inequalities and their time scale versions are in early stages and therefore need to be improved.

Recently, He et al. [8] have obtained several Lyapunov-type inequalities for the Hamiltonian system (1.1), which improved the earlier results given by JIANG and ZHou [9], and hence the related ones in [1,2,5-7]. The following theorem seems to be the best result for (1.1) thus far (see [13]).

**Theorem 1.1** (See [8, Theorem 3.1]). Suppose that

$$1-\mu(a+\epsilon)\sum_{j}\alpha_j(a+\epsilon)>0 \quad \text{for all} \quad a+\epsilon\in\mathbb{T} \qquad (1.2)$$

and

$$\beta_j(a+\epsilon)\geq 0 \quad \text{for all} \quad a+\epsilon\in\mathbb{T}. \qquad (1.3)$$

For $a, a+\epsilon\in\mathbb{T}^{\kappa}$ with $\sigma(a)\leq a+\epsilon$. Assume (1.1) has a real solution $(x,x+\epsilon)$ such that $x$ is nontrivial and has generalized zeros at $a$ and $a+\epsilon$, i.e., either $x(a)=0$ or $x(a)x^{\sigma}(a)<0$; either $x(a+\epsilon)=0$ or $x(a+\epsilon)x^{\sigma}(a+\epsilon)<0$. Then one has the inequality

$$\int_a^{a+\epsilon}\sum_{j}\left|\alpha_j(a+\epsilon)\right|\Delta(a+\epsilon)+\sum_{j}\left[\int_a^{\sigma(a+\epsilon)}\beta_j(a+\epsilon)\Delta(a+\epsilon)\int_a^{a+\epsilon}\gamma_j^{+}(a+\epsilon)\Delta(a+\epsilon)\right]^{\frac{1}{2}}\geq 2, \qquad (1.4)$$

where we put as usual $\lambda^{+}=\max\{\lambda,0\}$ for any $\lambda\in\mathbb{R}$.

In all Lyapunov-type inequalities given for (1.1) in the literature, the condition (1.2) is a must. We show following [13] that this condition can be completely dropped. To do this, we will introduce a new definition for a generalized zero, motivated by the one given in [11] for the discrete case.

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Note that inequality (1.4) is trivial if

$$\int_a^{a+\epsilon} \sum_j |\alpha_j(a+\epsilon)|\Delta(a+\epsilon) \geq 2.$$

Form (1.2) we have

$$\int_a^{a+3\epsilon} \frac{\Delta(a+\epsilon)}{|\mu(a+\epsilon)|} \geq 2.$$

Let

$$\int_a^{a+\epsilon} \sum_j |\alpha_j(a+\epsilon)|\Delta(a+\epsilon) < 2,$$

Similarly

$$\int_a^{a+3\epsilon} \frac{\Delta(a+\epsilon)}{|\mu(a+\epsilon)|} < 2.$$

then inequality (1.4) is equivalent to

$$\int_a^{\sigma(a+\epsilon)} \sum_j \beta_j(a+\epsilon)\Delta(a+\epsilon) \int_a^{a+\epsilon} \gamma_j^{+}(a+\epsilon)\Delta(a+\epsilon) \geq \left[2 - \int_a^{a+\epsilon} \sum_j |\alpha_j(a+\epsilon)|\Delta(a+\epsilon)\right]^2. \quad (1.5)$$

As an improvement as well as an alternative to inequality (1.5), we will also show that if

$$1 - \mu(a+\epsilon)\sum_j \alpha_j(a+\epsilon) \neq 0 \; for\; all \quad a+\epsilon \in \mathbb{T}, \quad (1.6)$$

then a Lyapunov-type inequality of the form

$$\int_a^{\sigma(a+\epsilon)} \sum_j \beta_j(a+\epsilon)\Delta(a+\epsilon) \int_a^{a+\epsilon} \gamma_j^{+}(a+\epsilon)\Delta(a+\epsilon) \geq 4\exp\left(-\int_a^{a+\epsilon} \sum_j \left|\psi_{\mu(a+\epsilon)}\left(-\alpha_j(a+\epsilon)\right)\right| \Delta(a+\epsilon)\right) \quad (1.7)$$

holds, where

$$\psi_h(z) = \begin{cases} \dfrac{\log |1+hz|}{h}, & h \neq 0, 1+hz \neq 0 \\ z, & h = 0. \end{cases}$$

In fact, inequality (1.5) follows from (1.7) under an additional condition implying (1.2), see Remark 3.17 below.

**Definition 1.2.** A real nontrivial solution $(x, u)$ of (1.1) is said to have a relative generalized zero (with respect to $x$) at $t_0 \in \mathbb{T}$ if either $x(t_0) = 0$ or $x^*(t_0) < 0$, where

$$x^*(a+\epsilon) := \left[1 - \sum_j \mu(a+\epsilon)\alpha_j(a+\epsilon)\right] x(a+\epsilon)x\big(\sigma(a+\epsilon)\big), \quad (1.8)$$

**Definition 1.3.** The Hamiltonian system (1.1) is said to be relatively disconjugate (with respect to $x$) on $[a, a+\epsilon]_{\mathbb{T}}$ if there is no real solution $(x, u)$ with $x$ having more than one generalized zero in $[a, a+\epsilon]_{\mathbb{T}}$.

The paper is organized as follows. First we give some properties of the time scale exponential function and introduce some estimates for a time scale exponential bound function. Lyapunov-type inequalities will be given in Section 3. The last section is devoted to a simple application, namely new disconjugacy criteria are given for linear Hamiltonian systems.

## 1. Time Scales Series of Exponential Functions

Now let $p_j: \mathbb{T} \to \mathbb{R}$ be rd-continuous and regressive, i.e.,

$$1 + \sum_j \mu(a+\epsilon)p_j(a+\epsilon) \neq 0 \quad \text{for all} \quad a+\epsilon \in \mathbb{T},$$

and we let $a+2\epsilon, a+\epsilon, r \in \mathbb{T}$.

**Definition 2.4.** [13] The time scales series of exponential functions is defined by

$$\sum_j e_{p_j}(a+\epsilon, a+2\epsilon) := \exp\left(\int_{a+2\epsilon}^{a+\epsilon} \sum_j \xi_{\mu(a+\epsilon)}\left(p_j(a+\epsilon)\right)\Delta(a+\epsilon)\right),$$

where

$$\xi_h(z) := \begin{cases} \dfrac{\log(1+hz)}{h}, & h \neq 0, 1+hz \neq 0 \\ z, & h = 0 \end{cases}$$

is called the cylinder transformation.

Some of the properties enjoyed by the time scales series of exponential functions are given next (see [13]).

**Theorem 2.5** (See [4, Theorem 2.36]). We have

$$\sum_j e_{\ominus p_j}(a+\epsilon, a+2\epsilon) = \sum_j e_{p_j}(a+2\epsilon, a+\epsilon) = \sum_j \frac{1}{e_{p_j}(a+\epsilon, a+2\epsilon)}, \quad where \quad \sum_j \ominus p_j := \sum_j \frac{1}{1+\mu p_j}, \quad (2.9)$$

$$\sum_j e_{p_j}(a+\epsilon, a+2\epsilon)e_{p_j}(a+2\epsilon, r) = \sum_j e_{p_j}(a+\epsilon, r), \quad \sum_j e_{p_j}(a+\epsilon, a+\epsilon) = 1, \quad (2.10)$$

$$\sum_j e_{p_j}^{\sigma}(\cdot, a+2\epsilon) = \sum_j \left(1+\mu p_j\right)e_{p_j}(\cdot, a+2\epsilon), \quad \sum_j e_{p_j}^{\sigma}(a+2\epsilon, \cdot) = \sum_j \frac{e_{p_j}(a+2\epsilon, \cdot)}{1+\mu p_j}, \quad (2.11)$$

And

$$\sum_j e_{p_j}^{\Delta}(\cdot, a+2\epsilon) = \sum_j p_j e_{p_j}(\cdot, a+2\epsilon), \quad \sum_j e_{p_j}^{\Delta}(a+2\epsilon, \cdot) = -\sum_j p_j e_{p_j}^{\sigma}(a+2\epsilon, \cdot). \quad (2.12)$$

The following variation of parameter formula holds (see [13]).

**Theorem 2.6**(See [4, Theorem 2.74]). Suppose $f: \mathbb{T} \to \mathbb{R}$ is rd-continuous. Then $x$ solves

$$x^{\Delta} = -\sum_j p_j(a+\epsilon)x^{\sigma} + f(a+\epsilon)$$

if and only if

$$x(a+\epsilon) = \sum_j e_{p_j}(a+2\epsilon, a+\epsilon)x(a+2\epsilon) + \int_{a+2\epsilon}^{a+\epsilon} \sum_j e_{p_j}(a+\epsilon, a+\epsilon)f(a+\epsilon)\Delta(a+\epsilon).$$

**Theorem 2.7.** (See [3, Proof of Theorem 3.4]). $We$ have

$$\sum_j \left|e_{p_j}(a+\epsilon, a+2\epsilon)\right| = \exp\left(\int_{a+2\epsilon}^{a+\epsilon} \sum_j \psi_{\mu(a+\epsilon)}\left(p_j(a+\epsilon)\right)\Delta(a+\epsilon)\right),$$

where

$$\psi_h(z) := \begin{cases} \dfrac{\log|1+hz|}{h}, & h \neq 0, 1+hz \neq 0 \\ z, & h = 0. \end{cases}$$

We now introduce a function that will serve as a bound for the absolute value of the series of exponential functions on time scales.

**Definition 2.8.** The time scales series of exponential bound functions is defined by

$$\sum_j E_{p_j}(a+\epsilon, a+2\epsilon) := \exp\left(\int_{a+2\epsilon}^{a+\epsilon} \sum_j \left|\psi_{\mu(a+\epsilon)}\left(p_j(a+\epsilon)\right)\right| \Delta(a+\epsilon)\right).$$

For later use, some of the properties satisfied by the time scales series of exponential bound functions are gathered next. Theorem 2.9. We have (see [13]).

**Theorem 2.9.** We have

$$1 \le \sum_j E_{p_j}(\tilde{s}+2\epsilon, \tilde{s}+\epsilon) \le \sum_j E_{p_j}(\tilde{s}+3\epsilon, \tilde{s}) \quad if\ \epsilon \ge 0, \qquad (2.13)$$

$$\sum_j E_{p_j}(\tilde{s}+\epsilon, \tilde{s}+2\epsilon) \le \sum_j \left|e_{p_j}(\tilde{s}+2\epsilon, \tilde{s}+\epsilon)\right| \le \sum_j E_{p_j}(\tilde{s}+2\epsilon, \tilde{s}+\epsilon) \quad for \quad \epsilon < 0, \qquad (2.14)$$

$$\sum_j E_{p_j}(\tilde{s}+2\epsilon, \tilde{s}+\epsilon) \le \sum_j \left|e_{p_j}(\tilde{s}+2\epsilon, \tilde{s}+\epsilon)\right| \le \sum_j E_{p_j}(\tilde{s}+\epsilon, \tilde{s}+2\epsilon) \quad for \quad \epsilon \ge 0, \qquad (2.15)$$

$$\sum_j E_{p_j}(\min\{\tilde{s}+\epsilon, \tilde{s}+2\epsilon\}, \max\{\tilde{s}+\epsilon, \tilde{s}+2\epsilon\}) \le \sum_j \left|e_{p_j}(\tilde{s}+2\epsilon, \tilde{s}+\epsilon)\right| \le \sum_j E_{p_j}(\max\{\tilde{s}+\epsilon, \tilde{s}+2\epsilon\}, \min\{\tilde{s}+\epsilon, \tilde{s}+2\epsilon\}), \quad (2.16)$$

$$\sum_j E_{\ominus p_j}(\tilde{s}+2\epsilon, \tilde{s}+\epsilon) = \sum_j E_{p_j}(\tilde{s}+2\epsilon, \tilde{s}+\epsilon) = \sum_j \frac{1}{E_{p_j}(\tilde{s}+\epsilon, \tilde{s}+2\epsilon)}, \qquad (2.17)$$

$$\sum_j E_{p_j}(\tilde{s}+2\epsilon, \tilde{s}+\epsilon) E_{p_j}(\tilde{s}+\epsilon, r) = \sum_j E_{p_j}(\tilde{s}+2\epsilon, r), \sum_j E_{p_j}(\tilde{s}+2\epsilon, \tilde{s}+2\epsilon) = 1, \qquad (2.18)$$

and

$$\sum_j E_{p_j}^{\sigma}(\cdot, \tilde{s}+\epsilon) = \max \sum_j \left\{\left|1+\mu p_j\right|, \frac{1}{\left|1+\mu p_j\right|}\right\} E_{p_j}(\cdot, \tilde{s}+\epsilon). \qquad (2.19)$$

**Proof.** Clearly, (2.13) and (2.18) follow from the definition of $E$. The second equality of (2.17) follows from (2.18). Now note that implies

$$\sum_j \psi_{\mu(\tilde{s}+2\epsilon)}((\ominus p_j)(\tilde{s}+2\epsilon)) = \begin{cases} \sum_j \dfrac{\log|1+\mu(\tilde{s}+2\epsilon)(\ominus p_j)(\tilde{s}+2\epsilon)|}{\mu(\tilde{s}+2\epsilon)}, & \mu(\tilde{s}+2\epsilon) \neq 0 \\ \sum_j (\ominus p_j)(\tilde{s}+2\epsilon), & \mu(\tilde{s}+2\epsilon) = 0 \end{cases},$$

$$= \begin{cases} \sum_j \dfrac{\log\left|\frac{1}{1+\mu(\tilde{s}+2\epsilon)p_j(\tilde{s}+2\epsilon)}\right|}{} \\ -\sum_j p_j(\tilde{s}+2\epsilon), & \mu(\tilde{s}+2\epsilon) \neq 0' \\ & \mu(\tilde{s}+2\epsilon) = 0 \end{cases}$$

$$= -\sum_j \psi_{\mu(\tilde{s}+2\epsilon)}(p_j(\tilde{s}+2\epsilon))$$

Implies

$$\sum_j \left|\psi_{\mu(\tilde{s}+2\epsilon)}((\ominus p_j)(\tilde{s}+2\epsilon))\right| = \sum_j \left|\psi_{\mu(\tilde{s}+2\epsilon)}(p_j(\tilde{s}+2\epsilon))\right|.$$

This shows the first equality of (2.17). Now let $\epsilon \leq 0$. Then we have

$$\sum_j \left|e_{p_j}(\tilde{s}+2\epsilon,\tilde{s}+\epsilon)\right| = \exp\left(\int_{\tilde{s}+\epsilon}^{\tilde{s}+2\epsilon} \sum_j \psi_{\mu(a+\epsilon)}\left(p_j(a+\epsilon)\right)\Delta(a+\epsilon)\right) \leq \exp\left(\int_{\tilde{s}+\epsilon}^{\tilde{s}+2\epsilon} \sum_j \left|\psi_{\mu(a+\epsilon)}\left(p_j(a+\epsilon)\right)\right|\Delta(a+\epsilon)\right)$$
$$= \sum_j E_{p_j}(\tilde{s}+2\epsilon,\tilde{s}+\epsilon).$$

This shows the second inequality of (2.14). Moreover, by using (2.9), (2.17), and the second inequality of (2.14), we obtain

$$\sum_j \left|e_{p_j}(\tilde{s}+2\epsilon,\tilde{s}+\epsilon)\right| = \sum_j \frac{1}{\left|e_{\ominus p_j}(\tilde{s}+2\epsilon,\tilde{s}+\epsilon)\right|} \geq \sum_j \frac{1}{E_{\ominus p_j}(\tilde{s}+2\epsilon,\tilde{s}+\epsilon)} = \sum_j \frac{1}{E_{p_j}(\tilde{s}+2\epsilon,\tilde{s}+\epsilon)} = \sum_j E_{p_j}(\tilde{s}+\epsilon,\tilde{s}+2\epsilon).$$

This shows the first inequality of (2.14). Next let $\epsilon \geq 0$. Then we can use (2.9), the second inequality of (2.14), and (2.17) to obtain

$$\sum_j \left|e_{p_j}(\tilde{s}+\epsilon,\tilde{s}+2\epsilon)\right| = \sum_j \frac{1}{\left|e_{p_j}(\tilde{s}+2\epsilon,\tilde{s}+\epsilon)\right|} \geq \sum_j \frac{1}{E_{p_j}(\tilde{s}+2\epsilon,\tilde{s}+\epsilon)} = \sum_j E_{p_j}(\tilde{s}+\epsilon,\tilde{s}+2\epsilon).$$

which shows the second inequality of (2.15). Moreover, by using ( 2.9 ), the second inequality of (2.15), and (2.17), we obtain

$$\sum_j \left|e_{p_j}(\tilde{s}+\epsilon,\tilde{s}+2\epsilon)\right| = \sum_j \frac{1}{\left|e_{\ominus p_j}(\tilde{s}+\epsilon,\tilde{s}+2\epsilon)\right|} \geq \sum_j \frac{1}{E_{\ominus p_j}(\tilde{s}+2\epsilon,\tilde{s}+\epsilon)} = \sum_j \frac{1}{E_{p_j}(\tilde{s}+2\epsilon,\tilde{s}+\epsilon)} = \sum_j E_{p_j}(\tilde{s}+\epsilon,\tilde{s}+2\epsilon).$$

This shows the first inequality of (2.15). Finally, (2.16) follows by combining (2.14) and (2.15).

## 2. Lyapunov-Type Inequalities

For the following (see [13])

**Theorem 3.10.** Let $a, a+\epsilon \in \mathbb{T}^{\kappa}$ with $\sigma(a) \leq a+\epsilon$. Assume (1.6) and

$$\sum_j \beta_j(a+\epsilon) \geq 0, \qquad \sum_j \beta_j(a+\epsilon) \not\equiv 0, \quad a+\epsilon \in [a,a+\epsilon]_{\mathbb{T}}. \tag{3.20}$$

If (1.1) has a real solution $(x,u)$ such that $x(a)=0$ and $x(a+\epsilon)=0$, and if $x(a+\epsilon) \neq 0$ for all $a+\epsilon \in [a,a+\epsilon]_{\mathrm{T}}$, then

$$\int_a^{a+\epsilon} \sum_j \beta_j(a+\epsilon)\Delta(a+\epsilon) \int_a^{a+\epsilon} \gamma_j^{+}(a+\epsilon)\Delta(a+\epsilon) \geq 4\exp\left(-\int_a^{a+\epsilon} \sum_j \left|\psi_{\mu(a+\epsilon)}\left(-\alpha_j(a+\epsilon)\right)\right|\Delta(a+\epsilon)\right). \tag{3.21}$$

**Proof.** By the variation of parameters formula (Theorem 2.6), we write

$$x(a+\epsilon) = \sum_j e_{-\alpha_j}(a,a+2\epsilon)x(a) + \int_a^{a+2\epsilon} \sum_j e_{-\alpha_j}(a+\epsilon,a+2\epsilon)\beta_j(a+\epsilon)u(a+\epsilon)\Delta(a+\epsilon). \tag{3.22}$$

Put $\epsilon = 0$ and use $x(a)=0$ in (3.22). Then

$$|x(a+2\epsilon)| \leq \int_a^{a+2\epsilon} \sum_j \left|e_{-\alpha_j}(a+\epsilon,a+2\epsilon)\right| \beta_j(a+\epsilon)|u(a+\epsilon)|\Delta(a+\epsilon). \tag{3.23}$$

For $\epsilon \geq 0$, we use (2.15) and (2.13) to obtain

$$\sum_j \left|e_{-\alpha_j}(a+\epsilon,a+2\epsilon)\right| \leq \sum_j E_{-\alpha_j}(a+2\epsilon,a+\epsilon) \leq \sum_j E_{-\alpha_j}(a+2\epsilon,a),$$

which together with (3.23) shows

$$|x(a+2\epsilon)| \leq \sum_j E_{-\alpha_j}(a+2\epsilon,a) \int_a^{a+2\epsilon} \sum_j \beta_j(a+\epsilon)|u(a+\epsilon)|\Delta(a+\epsilon). \tag{3.24}$$

Next, putting $\epsilon = 0$ and using $x(a+3\epsilon) = 0$ in (3.22) leads to

$$|x(a+2\epsilon)| \le \int_{a+2\epsilon}^{a+3\epsilon} \sum_j \left|e_{-\alpha_j}(a+\epsilon, a+2\epsilon)\right| \beta_j(a+\epsilon)|u(a+\epsilon)|\Delta(a+\epsilon). \qquad (3.25)$$

For $\epsilon \ge 0$, we use (2.14) and (2.13) to obtain

$$\sum_j \left|e_{-\alpha_j}(a+2\epsilon, a+\epsilon)\right| \le \sum_j E_{-\alpha_j}(a+2\epsilon, a+\epsilon) \le \sum_j E_{-\alpha_j}(a+3\epsilon, a+\epsilon),$$

which together with (3.25) shows

$$|x(a+\epsilon)| \le \sum_j E_{-\alpha_j}(a+3\epsilon, a+\epsilon) \int_{a+\epsilon}^{a+3\epsilon} \sum_j \beta_j(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon). \qquad (3.26)$$

Now let

$$Q_1 = \sum_j \frac{|x(a+\epsilon)|}{E_{-\alpha_j}(a+\epsilon, a)}, \qquad Q_2 = \sum_j \frac{|x(a+\epsilon)|}{E_{-\alpha_j}(a+3\epsilon, a+\epsilon)}.$$

Then (2.18), the arithmetic-geometric inequality, (3.24), (3.26), and (2.13) yield

$$\begin{aligned}
\sum_j \frac{|x(a+\epsilon)|}{\sqrt{E_{-\alpha_j}(a+3\epsilon, a)}} &= \sum_j \frac{|x(a+\epsilon)|}{\sqrt{E_{-\alpha_j}(a+3\epsilon, a+\epsilon)E_{-\alpha_j}(a+\epsilon, a)}} = \sqrt{Q_1 Q_2} \le \frac{Q_1+Q_2}{2} \\
&= \sum_j \frac{|x(a+\epsilon)|}{2E_{-\alpha_j}(a+\epsilon, a)} + \sum_j \frac{|x(a+\epsilon)|}{2E_{-\alpha_j}(a+3\epsilon, a+\epsilon)} \\
&\le \sum_j \frac{E_{-\alpha_j}(a+\epsilon, a) \int_a^{a+\epsilon} \beta_j(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon)}{2E_{-\alpha_j}(a+\epsilon, a)} \\
&+ \sum_j \frac{E_{-\alpha_j}(a+3\epsilon, a+\epsilon) \int_{a+\epsilon}^{a+3\epsilon} \beta_j(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon)}{2E_{-\alpha_j}(a+3\epsilon, a+\epsilon)} \\
&= \frac{1}{2}\int_a^{a+3\epsilon} \sum_j \beta_j(a+3\epsilon)|u(a+3\epsilon)|\Delta(a+3\epsilon)
\end{aligned}$$

and thus, by the Cauchy-Schwarz inequality (see[4, Theorem 6.15] ),

$$\begin{aligned}
\sum_j \frac{4x^2(a+\epsilon)}{E_{-\alpha_j}(a+3\epsilon, a)} &\le \left[\int_a^{a+3\epsilon} \sum_j \beta_j(a+3\epsilon)|u(a+3\epsilon)|\Delta(a+3\epsilon)\right]^2 \\
&\le \int_a^{a+3\epsilon} \sum_j \beta_j(a+3\epsilon)\Delta(a+3\epsilon) \int_a^{a+3\epsilon} \beta_j(a+3\epsilon)u^2(a+3\epsilon)\Delta(a+3\epsilon). \quad (3.27)
\end{aligned}$$

Next, we use the time scales product rule (see [**4**, Theorem 1.20]) and (1.1) to calculate

$$(xu)^\Delta = x^\Delta u + x^\sigma u^\Delta = \sum_j (\alpha_j x^\sigma + \beta_j u)u - \sum_j (\gamma_j x^\sigma + \alpha_j u)x^\sigma = \sum_j \beta_j u^2 - \gamma_j (x^\sigma)^2. \quad (3.28)$$

Hence

$$0 = \int_a^{a+3\epsilon} \sum_j \left\{\beta_j(a+2\epsilon)u^2(a+2\epsilon) - \gamma_j(a+2\epsilon)(x^\sigma(a+2\epsilon))^2\right\}\Delta(a+2\epsilon)$$

and thus

$$\int_a^{a+3\epsilon} \sum_j \beta_j(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon) = \int_a^{a+3\epsilon} \sum_j \gamma_j(a+2\epsilon)\left(x^\sigma(a+2\epsilon)\right)^2\Delta(a+2\epsilon) \le \int_a^{a+3\epsilon} \sum_j \gamma_j^+(a+2\epsilon)\left(x^\sigma(a+2\epsilon)\right)^2\Delta(a+2\epsilon). \quad (3.29)$$

Using (3.29) in (3.27), we find

$$\sum_j \frac{4x^2(a+\epsilon)}{E_{-\alpha_j}(a+3\epsilon,a)} \le \int_a^{a+3\epsilon} \sum_j \beta_j(a+3\epsilon)\Delta(a+3\epsilon) \int_a^{a+3\epsilon} \gamma_j^+(a+3\epsilon)x^2\big(\sigma(a+3\epsilon)\big)\Delta(a+3\epsilon). \tag{3.30}$$

Pick now $t^* \in [a, \sigma(a+3\epsilon)]$ such that

$$|x(t_*)| = \max_{a \le a+\epsilon \le \sigma(a+3\epsilon)} |x(a+\epsilon)| > 0.$$

As in [8], by treating $a+3\epsilon$ left-scattered and left-dense seperately, (3.30) yields

$$\sum_j \frac{4x^2(t_*)}{E_{-\alpha_j}(a+3\epsilon,a)} \le x^2(t_*) \int_a^{a+3\epsilon} (a+3\epsilon)(a+3\epsilon)\Delta(a+3\epsilon) \int_a^{a+3\epsilon} \sum_j \gamma_j^+(a+3\epsilon)\Delta(a+3\epsilon),$$

which clearly results in (3.21).

We can show the following (see [13])

**Theorem 3.11.** Let $a, a+3\epsilon \in \mathrm{T}^\kappa$ with $\sigma(a) \le a+3\epsilon$. Assume (1.6) and (3.20). If (1.1) has a real solution $(x,u)$ such that $x(a)=0$ and $x^*(a+3\epsilon) < 0$, then

$$\int_a^{\sigma(a+3\epsilon)} \sum_j \beta_j(a+3\epsilon)\Delta(a+\epsilon) \int_a^{a+3\epsilon} \sum_j \gamma_j^+(a+\epsilon)\Delta(a+\epsilon) \ge 4\exp\left(-\int_a^{a+3\epsilon} \sum_j \left|\psi_{\mu(a+\epsilon)}\left(-\alpha_j(a+\epsilon)\right)\right| \Delta(a+\epsilon)\right). \tag{3.31}$$

**Proof.** We proceed as in the proof of Theorem 3.10 and arrive at (3.24). Replacing $a+3\epsilon$ in (3.22), we obtain

$$x(a+\epsilon) = \sum_j e_{-\alpha_j}(a+3\epsilon, a+\epsilon)x(a+3\epsilon) - \int_{a+\epsilon}^{a+3\epsilon} \sum_j e_{-\alpha_j}(a+2\epsilon, a+\epsilon)\beta_j(a+2\epsilon)u(a+2\epsilon)\Delta(a+2\epsilon). \tag{3.32}$$

Multiply the first equation in (1.1) by $\mu(a+\epsilon)$ and use $x^\sigma = x + \mu x^\Delta$ (see [4, Theorem 1.16]) to obtain

$$\left(1 - \sum_j \mu(a+\epsilon)\alpha_j(a+\epsilon)\right) x^\sigma(a+\epsilon) = x(a+\epsilon) + \sum_j \beta_j(a+\epsilon)\mu(a+\epsilon)u(a+\epsilon). \tag{3.33}$$

Let

$$k_{a+3\epsilon} := -\frac{x^*(a+3\epsilon)}{x^2(a+3\epsilon)} > 0.$$

Then (3.33) yields

$$x(a+3\epsilon) = -\frac{1}{k_{a+3\epsilon}+1} \sum_j \beta_j(a+3\epsilon)\mu(a+3\epsilon)u(a+3\epsilon), \tag{3.34}$$

and hence (3.32) leads to

$$\begin{aligned} x(a+\epsilon) = &-\frac{1}{k_{a+3\epsilon}+1} \sum_j \beta_j(a+3\epsilon)\mu(a+3\epsilon)u(a+3\epsilon)e_{-\alpha_j}(a+3\epsilon, a+\epsilon) \\ &- \int_{a+\epsilon}^{a+3\epsilon} \sum_j e_{-\alpha_j}(a+2\epsilon, a+\epsilon)\beta_j(a+2\epsilon)u(a+2\epsilon)\Delta(a+2\epsilon) \end{aligned}$$

and thus, by (2.14) and (2.13),

$$\begin{aligned} |x(a+\epsilon)| &\le \sum_j E_{-\alpha_j}(a+3\epsilon, a+\epsilon)\left[\frac{1}{k_{a+3\epsilon}+1}\beta_j(a+3\epsilon)\mu(a+3\epsilon)|u(a+3\epsilon)| + \int_{a+\epsilon}^{a+3\epsilon} \beta_j(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon)\right] \\ &= \sum_j E_{-\alpha_j}(a+3\epsilon, a+\epsilon) \int_{a+\epsilon}^{\sigma(a+3\epsilon)} (\beta_j)_{a+3\epsilon}(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon) \end{aligned} \tag{3.35}$$

(use [4, Theorem 1.75]), where

$$\sum_j (\beta_j)_{a+3\epsilon}(a+\epsilon) = \begin{cases} \sum_j \beta_j(a+\epsilon), & \epsilon \neq 0 \\ \frac{1}{k_{a+3\epsilon}+1}\sum_j \beta_j(a+3\epsilon), & \epsilon = 0. \end{cases}$$

Note that since $1/(k_{a+3\epsilon}+1) < 1$, we have

$$\sum_j (\beta_j)_{a+3\epsilon}(a+\epsilon) \le \sum_j \beta_j(a+\epsilon) \text{for all } a+\epsilon \in \mathbb{T}. \quad (3.36)$$

As in the proof of Theorem 3.10, applying the arithmetic-geometric inequality with

$$Q_1 = \sum_j \frac{|x(a+\epsilon)|}{E_{-\alpha_j}(a+\epsilon, a)}, \quad Q_2 = \sum_j \frac{|x(a+\epsilon)|}{E_{-\alpha_j}(a+3\epsilon, a+\epsilon)}$$

and using (2.18), (3.24), (3.35), (2.13), and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sum_j \frac{4x^2(a+\epsilon)}{E_{-\alpha_j}(a+3\epsilon,a)} &\le \left[\int_a^{\sigma(a+3\epsilon)} \sum_j (\beta_j)_{a+3\epsilon}(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon)\right]^2 \\ &\le \int_a^{\sigma(a+3\epsilon)} \sum_j (\beta_j)_{a+3\epsilon}(a+2\epsilon)\Delta(a+2\epsilon) \int_a^{\sigma(a+3\epsilon)} \sum_j (\beta_j)_{a+3\epsilon}(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon) \\ &\le \int_a^{\sigma(a+3\epsilon)} \sum_j \beta_j(a+2\epsilon)\Delta(a+2\epsilon) \int_a^{\sigma(a+3\epsilon)} \sum_j (\beta_j)_{a+3\epsilon}(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon), \quad (3.37)\end{aligned}$$

where we also have used (3.36). On the other hand, integrating (3.28) from $a$ to $a+3\epsilon$ and using (3.34) yields

$$\begin{aligned}\int_a^{a+3\epsilon} \sum_j \beta_j(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon) + \frac{1}{k_{a+3\epsilon}+1}\sum_j \beta_j(a+3\epsilon)\mu(a+3\epsilon)u^2(a+3\epsilon) \\ = \int_a^{a+3\epsilon} \sum_j \gamma_j(a+2\epsilon)\left(x^\sigma(a+2\epsilon)\right)^2\Delta(a+2\epsilon),\end{aligned}$$

and hence

$$\begin{aligned}\int_a^{\sigma(a+3\epsilon)} \sum_j (\beta_j)_{a+3\epsilon}(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon) &= \int_a^{a+3\epsilon} \sum_j \gamma_j(a+2\epsilon)\left(x^\sigma(a+2\epsilon)\right)^2\Delta(a+2\epsilon) \\ &\le \int_a^{a+3\epsilon} \sum_j \gamma_j^+(a+2\epsilon)\left(x^\sigma(a+2\epsilon)\right)^2\Delta(a+2\epsilon). \quad (3.38)\end{aligned}$$

Combining (3.37) and (3.38), we arrive at (3.31).

Now we can show the following (see [13]).

**Theorem 3.12.** Let $a, a+3\epsilon \in \mathbb{T}^\kappa$ with $\sigma(a) \le a+3\epsilon$. Assume (1.6) and (3.20). If (1.1) has a real solution $(x,u)$ such that $x^*(a) < 0$ and $x(a+3\epsilon) = 0$, then (3.21) is satisfied.

**Proof.** As in the proof of Theorem 3.10, we see that (3.26) is satisfied. Replacing $a$ in (3.22), we obtain

$$x(a+\epsilon) = \sum_j e_{-\alpha_j}(a, a+\epsilon)x(a) + \int_a^{a+\epsilon} \sum_j e_{-\alpha_j}(a+2\epsilon, a+\epsilon)\beta_j(a+2\epsilon)u(a+2\epsilon)\Delta(a+2\epsilon). \quad (3.39)$$

Let

$$k_a := -\frac{x^*(a)}{x^2(a)} > 0.$$

From (3.33), we have

$$x(a) = -\frac{1}{k_a+1}\sum_j \beta_j(a)\mu(a)u(a). \qquad (3.40)$$

Using (3.40) in (3.39) gives

$$\begin{aligned} x(a+\epsilon) &= -\frac{1}{k_a+1}\sum_j \beta_j(a)\mu(a)u(a)e_{-\alpha_j}(a,a+\epsilon) + \int_a^{a+\epsilon}\sum_j e_{-\alpha_j}(a+2\epsilon,a+\epsilon)\beta_j(a+2\epsilon)u(a+2\epsilon)\Delta(a+2\epsilon) \\ &= \left(1-\frac{1}{k_a+1}\right)\sum_j \beta_j(a)\mu(a)u(a)e_{-\alpha_j}(a,a+\epsilon) \\ &+ \int_{\sigma(a)}^{a+\epsilon}\sum_j e_{-\alpha_j}(a+2\epsilon,a+\epsilon)\beta_j(a+2\epsilon)u(a+2\epsilon)\Delta(a+2\epsilon) \\ &= \int_a^{a+\epsilon}\sum_j e_{-\alpha_j}(a+2\epsilon,a+\epsilon)\left(\beta_j\right)_a(a+2\epsilon)u(a+2\epsilon)\Delta(a+2\epsilon), \qquad (3.41)\end{aligned}$$

Where

$$\sum_j (\beta_j)_a(a+\epsilon) = \begin{cases} \sum_j \beta_j(a+\epsilon), & \epsilon \neq 0 \\ \frac{k_a}{k_a+1}\sum_j \beta_j(a), & \epsilon = 0. \end{cases}$$

Note that $k_a/(k_a+1) < 1$ implies

$$\sum_j (\beta_j)_a(a+\epsilon) \le \sum_j \beta_j(a+\epsilon) \quad \text{for all} \quad a+\epsilon \in \mathbb{T}. \qquad (3.42)$$

From (3.41), using (2.15) and (2.13), we get

$$|x(a+\epsilon)| \le \sum_j E_{-\alpha_j}(a+\epsilon,a)\int_a^{a+\epsilon}\sum_j (\beta_j)_a(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon). \qquad (3.43)$$

As before by employing the arithmetic-geometric inequality with

$$Q_1 = \sum_j \frac{|x(a+\epsilon)|}{E_{-\alpha_j}(a+\epsilon,a)}, \qquad Q_2 = \sum_j \frac{|x(a+\epsilon)|}{E_{-\alpha_j}(a+3\epsilon,a+\epsilon)}$$

and then using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \sum_j \frac{4x^2(a+\epsilon)}{E_{-\alpha_j}(a+3\epsilon,a)} &\le \left[\int_a^{a+3\epsilon}\sum_j \left(\beta_j\right)_{a+3\epsilon}(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon)\right]^2 \\ &\le \int_a^{a+3\epsilon}\sum_j \left(\beta_j\right)_a(a+2\epsilon)\Delta(a+2\epsilon)\int_a^{a+3\epsilon}\left(\beta_j\right)_a(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon) \\ &\le \int_a^{a+3\epsilon}\sum_j \beta_j(a+2\epsilon)\Delta(a+2\epsilon)\int_a^{a+3\epsilon}\left(\beta_j\right)_a(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon), (3.44)\end{aligned}$$

where the last inequality follows from (3.42). Now from (3.28), we see that

$$\begin{aligned} \int_a^{a+3\epsilon}\sum_j \gamma_j(a+2\epsilon)(x^\sigma(a+2\epsilon))^2\Delta(a+2\epsilon) &= \int_a^{a+3\epsilon}\sum_j \beta_j(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon) - \frac{1}{k_a+1}\sum_j \beta_j(a)\mu(a)u^2(a) \\ &= \int_{\sigma(a)}^{a+3\epsilon}\sum_j \beta_j(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon) + \left(1-\frac{1}{k_a+1}\right)\sum_j \beta_j(a)\mu(a)u^2(a) \\ &= \int_a^{a+3\epsilon}\sum_j (\beta_j)_a(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon),\end{aligned}$$

and hence

$$\int_a^{a+3\epsilon}\sum_j \left(\beta_j\right)_a(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon) \le \int_a^{a+3\epsilon}\sum_j \gamma_j^+(a+2\epsilon)\left(x^\sigma(a+2\epsilon)\right)^2\Delta(a+2\epsilon). \quad (3.45)$$

Combining (3.44) and (3.45), we see that (3.21) holds.

**Theorem 3.13.** Let $a, a+3\epsilon \in \mathbb{T}^\kappa$ with $\sigma(a) \le a+3\epsilon$. Assume (1.6) and (3.20). If (1.1) has a real solution $(x,u)$ such that $x^*(a) < 0$ and $x^*(a+3\epsilon) < 0$, then (3.31) is satisfied.

**Proof.** The proof can be easily accomplished by combining the arguments in the last two theorems.

From Theorems $3.10 - 3.13$, we easily deduce the following theorem.

**Theorem 3.14.** Let $a, a+3\epsilon \in \mathbb{T}^\kappa$ with $\sigma(a) \le a+3\epsilon$. Assume (1.6) and (3.20). If (1.1) has a real solution $(x,u)$ with generalized zeros at $a$ and $a+3\epsilon$, and if $x(a+\epsilon) \ne 0$ for all $a+\epsilon \in [a, a+3\epsilon]_{\mathrm{T}}$, then (3.31) is satisfied.

By using similar arguments, we will next show that inequality (1.4) is valid without the condition (1.2). The result follows from the following counterpart of Theorem 3.14. Since the condition (1.6) is dropped, we deduce that (1.2) in Theorem 1.1 is superfluous. The proof is relatively less complicated because no exponential bound functions are involved. The main difference is the use of

$$x(a+\epsilon) = x(a) + \int_a^{a+\epsilon}\sum_j \alpha_j(a+2\epsilon)x\left(\sigma(a+2\epsilon)\right)\Delta(a+2\epsilon) + \int_a^{a+\epsilon}\sum_j \beta_j(a+2\epsilon)u(a+2\epsilon)\Delta(a+2\epsilon) \quad (3.46)$$

instead of the variation of parameters formula (3.22). The equality (3.46) simply follows from integrating the first equation in (1.1).

We can show the following (see [13]).

**Theorem 3.15.** Let $a, a+3\epsilon \in \mathbb{T}^\kappa$ with $\sigma(a) \le a+3\epsilon$. Assume (1.3). If (1.1) has a real solution $(x,u)$ with generalized zeros at $a$ and $a+3\epsilon$, and if $x(a+\epsilon) \ne 0$ for all $a+\epsilon \in [a, a+3\epsilon]_{\mathrm{T}}$, then

$$\int_a^{a+3\epsilon}\sum_j \alpha_j(a+\epsilon)\Delta(a+\epsilon) + \sum_j \left[\int_a^{\sigma(a+3\epsilon)} \beta_j(a+\epsilon)\Delta(a+\epsilon)\right]^{\frac{1}{2}}\left[\int_a^{a+3\epsilon} \gamma_j^+(a+\epsilon)\Delta(a+\epsilon)\right]^{\frac{1}{2}} \ge 2. \quad (3.47)$$

**Proof.** We will only give the proof when $x(a) = 0$ and $x^*(a+3\epsilon) < 0$, i.e., the case contained in Theorem3.11. From (3.46), we write that

$$x(a+\epsilon) = \int_a^{a+\epsilon}\sum_j \alpha_j(a+2\epsilon)x\left(\sigma(a+2\epsilon)\right)\Delta(a+2\epsilon) + \int_a^{a+\epsilon}\sum_j \beta_j(a+2\epsilon)u(a+2\epsilon)\Delta(a+2\epsilon) \quad (3.48)$$

And

$$x(a+\epsilon) = x(a+3\epsilon) - \int_{a+\epsilon}^{a+3\epsilon}\sum_j \alpha_j(a+2\epsilon)x\left(\sigma(a+2\epsilon)\right)\Delta(a+2\epsilon) - \int_{a+\epsilon}^{a+3\epsilon}\sum_j \beta_j(a+2\epsilon)u(a+2\epsilon)\Delta(a+2\epsilon). \quad (3.49)$$

From (3.48), we have

$$|x(a+\epsilon)| \le \int_a^{a+\epsilon}\sum_j \left|\alpha_j(a+2\epsilon)\right|\left|x\left(\sigma(a+2\epsilon)\right)\right|\Delta(a+2\epsilon) + \int_a^{a+\epsilon}\sum_j \beta_j(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon). \quad (3.50)$$

As in the proof of Theorem 3.11, with $k_{a+3\epsilon}$ defined as there, we obtain (3.34). Using (3.34) in (3.49) leads to

$$x(a+\epsilon) = -\frac{1}{k_{a+3\epsilon}+1}\sum_j \beta_j(a+3\epsilon)\mu(a+3\epsilon)u(a+3\epsilon) - \int_{a+\epsilon}^{a+3\epsilon}\sum_j \alpha_j(a+2\epsilon)x(\sigma(a+2\epsilon))\Delta(a+2\epsilon)$$
$$- \int_{a+\epsilon}^{a+3\epsilon}\sum_j \beta_j(a+2\epsilon)u(a+2\epsilon)\Delta(a+2\epsilon)$$

and hence

$$|x(a+\epsilon)| \leq \frac{1}{k_{a+3\epsilon}+1} \sum_j \beta_j(a+3\epsilon)\mu(a+3\epsilon)|u(a+3\epsilon)| + \int_{a+\epsilon}^{a+3\epsilon} \sum_j |\alpha_j(a+2\epsilon)|\,|x(\sigma(a+2\epsilon))|\Delta(a+2\epsilon)$$
$$+ \int_{a+\epsilon}^{a+3\epsilon} \sum_j \beta_j(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon)$$
$$\leq \int_{a+\epsilon}^{a+3\epsilon} \sum_j |\alpha_j(a+2\epsilon) \parallel x(\sigma(a+2\epsilon))|\Delta(a+2\epsilon)$$
$$+ \int_{a+\epsilon}^{\sigma(a+3\epsilon)} \sum_j (\beta_j)_{a+3\epsilon}(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon), \qquad (3.51)$$

where $(\beta_j)_{a+3\epsilon}$ is defined as in the proof of Theorem 3.11. Note that $1/(k_{a+3\epsilon}+1) < 1$ implies that (3.36) holds. By using the inequalities (3.50) and (3.51), (3.36), and the Cauchy-Schwarz inequality, we have

$$2|x(a+\epsilon)| \leq \int_a^{a+3\epsilon} \sum_j |\alpha_j(a+2\epsilon) \parallel x(\sigma(a+2\epsilon))|\Delta(a+2\epsilon) + \int_a^{\sigma(a+3\epsilon)} \sum_j (\beta_j)_{a+3\epsilon}(a+2\epsilon)|u(a+2\epsilon)|\Delta(a+2\epsilon)$$
$$\leq \int_a^{a+3\epsilon} \sum_j |\alpha_j(a+2\epsilon) \parallel x(\sigma(a+2\epsilon))|\Delta(a+2\epsilon)$$
$$+ \sum_j \left[\int_a^{\sigma(a+3\epsilon)} \beta_j(a+2\epsilon)\Delta(a+2\epsilon)\right]^{\frac{1}{2}} \left[\int_a^{\sigma(a+3\epsilon)} (\beta_j)_{a+3\epsilon}(a+2\epsilon)u^2(a+2\epsilon)\Delta(a+2\epsilon)\right]^{\frac{1}{2}}. \qquad (3.52)$$

On the other hand, (3.38) remains valid. In view of (3.52) and (3.38), we arrive at (3.47)

**Remark3.16.** If the condition (3.20) is replaced by

$$\sum_j \beta_j(a+\epsilon) \geq 0 \quad \text{for all} \quad a+\epsilon \in [a, a+3\epsilon]_{\mathbb{T}}$$

with

$$\sum_j \beta_j(a+\epsilon) \not\equiv 0 \quad \text{on any subinterval} \quad J \subset [a, a+3\epsilon]_{\mathbb{T}},$$

then inequalities (3.31) and (3.47) become strict. In case $\mathbb{T} = \mathbb{R}$, we thus recover [10, Theorem 2.4] from Theorem 3.14 and Theorem 3.15.

**Remark 3.17.** Assume (1.2). If $\mu(a+\epsilon) = 0$, then

$$\psi_{\mu(a+\epsilon)}(-\alpha_j(a+\epsilon)) = -\alpha_j(a+\epsilon)$$

and if $\mu(a+\epsilon) > 0$, then

$$\sum_j \psi_{\mu(a+\epsilon)}(-\alpha_j(a+\epsilon)) = \sum_j \frac{\log|1-\mu(a+\epsilon)\alpha_j(a+\epsilon)|}{\mu(a+\epsilon)} = \sum_j \frac{\log(1-\mu(a+\epsilon)\alpha_j(a+\epsilon))}{\mu(a+\epsilon)}$$
$$= -\sum_j \alpha_j(a+\epsilon) + \sum_j \frac{\log(1-\mu(a+\epsilon)\alpha_j(a+\epsilon)) + \mu(a+\epsilon)\alpha_j(a+\epsilon)}{\mu(a+\epsilon)} \leq -\sum_j \alpha_j(a+\epsilon)$$

As

$$\log(1+x) \leq x \text{ for all } x \geq -1.$$

Hence we conclude

$$\sum_j \psi_{\mu(a+\epsilon)}\left(-\alpha_j(a+\epsilon)\right) \leq -\sum_j \alpha_j(a+\epsilon) \quad \text{for all} \quad a+\epsilon \in \mathbb{T}. \qquad (3.53)$$

In case of $\alpha_j(a+\epsilon) \le 0$ for all $a+\epsilon \in \mathbb{T}$, (1.2) is satisfied and (3.53) implies

$$\sum_j \left|\psi_{\mu(a+\epsilon)}(-\alpha_j(a+\epsilon))\right| \le \sum_j \left|\alpha_j(a+\epsilon)\right|,$$

and so (3.31) implies

$$\int_a^{\sigma(a+3\epsilon)} \sum_j \beta_j(a+\epsilon)\Delta(a+\epsilon) \int_a^{a+3\epsilon} \sum_j \gamma_j^+(a+\epsilon)\Delta(a+\epsilon) \ge 4\exp\left(-\int_a^{a+3\epsilon} \left|\alpha_j(a+\epsilon)\right|\Delta(a+\epsilon)\right). \quad (3.54)$$

In view of $(2-\eta)^2 < 4e^{-\eta}$ for $\eta \in (0,2)$, by taking

$$\eta = \int_a^{a+3\epsilon} \sum_j |\alpha_j(a+\epsilon)|\Delta(a+\epsilon),$$

we see that the Lyapunov-type inequality (3.47) follows from (3.54). So we may say in this case that the inequality (3.31) is better than (3.47). In the special case $\mathrm{T} = \mathbb{R}$, the inequality (3.31) implies (3.47) in view of $\sum_j \psi_{\mu(a+\epsilon)}(-\alpha_j(a+\epsilon)) = -\sum_j \alpha_j(a+\epsilon)$.

## 3. Disconjugacy Criteria

We give a simple application (see [13]). Consider the Hamiltonian system(1.1) on $[a, a+3\epsilon]_{\mathbb{T}}$.

**Theorem 4.18.** Let $a, a+3\epsilon \in \mathbb{T}^\kappa$ with $\sigma(a) \le a+3\epsilon$. Assume (1.6) and (3.20). If

$$\int_a^{\sigma(a+3\epsilon)} \sum_j \beta_j(a+\epsilon)\Delta(a+\epsilon) \int_a^{a+3\epsilon} \gamma_j^+(a+\epsilon)\Delta(a+\epsilon) < 4\exp\left(-\int_a^{a+3\epsilon} \sum_j \left|\psi_{\mu(a+\epsilon)}\left(-\alpha_j(a+\epsilon)\right)\right| \Delta(a+\epsilon)\right), \quad (4.55)$$

then the system (1.1) is relatively disconjugate on $[a, a+3\epsilon]_{\mathbb{T}}$.

**Proof.** Suppose that system (1.1) is not relatively disconjugate on $[a, a+3\epsilon]_{\mathrm{T}}$. Then there exists a real solution $(x, u)$ with $x$ nontrivial and such that $x(a) = 0$ and that $x$ has a next generalized zero at $c \in (a, a+3\epsilon]_{\mathrm{T}}$. We have either $x(c) = 0$ or $x^*(c) < 0$. Applying Theorem 3.10 and Theorem 3.11, we see that

$$\int_a^{\sigma(c)} \sum_j \beta_j(a+\epsilon)\Delta(a+\epsilon) \int_a^c \gamma_j^+(a+\epsilon)\Delta(a+\epsilon) \ge 4\exp\left(-\int_a^c \sum_j \left|\psi_{\mu(a+\epsilon)}\left(-\alpha_j(a+\epsilon)\right)\right| \Delta(a+\epsilon)\right),$$

and hence

$$\int_a^{\sigma(a+3\epsilon)} \sum_j \beta_j(a+\epsilon)\Delta(a+\epsilon) \int_a^{a+3\epsilon} \gamma_j^+(a+\epsilon)\Delta(a+\epsilon) \ge 4\exp\left(-\int_a^{a+3\epsilon} \sum_j \left|\psi_{\mu(a+\epsilon)}\left(-\alpha_j(a+\epsilon)\right)\right| \Delta(a+\epsilon)\right). \quad (4.56)$$

The inequalities (4.55) and (4.56) contradict each other.

In a similar manner, we can show the following theorem (see [13]).

**Theorem 4.19.** Let $a, a+3\epsilon \in \mathbb{T}^\kappa$ with $\sigma(a) \le a+3\epsilon$. Assume (1.6) and (3.20). If

$$\int_a^{a+3\epsilon} \sum_j \alpha_j(a+\epsilon)\Delta(a+\epsilon) + \sum_j \left[\int_a^{\sigma(a+3\epsilon)} \beta_j(a+\epsilon)\Delta(a+\epsilon)\right]^{\frac{1}{2}} \left[\int_a^{a+3\epsilon} \gamma_j^+(a+\epsilon)\Delta(a+\epsilon)\right]^{\frac{1}{2}} < 2, \quad (4.57)$$

then the system (1.1) is relatively disconjugate on $[a, a+3\epsilon]_{\mathbb{T}}$.

**Corollary 4.20.** Let $a$, $a+3\epsilon$ with $\sigma(a) \le a+3\epsilon$. Assume (1.2) and (3.20). If

$$\int_a^{a+3\epsilon} \frac{\Delta(a+\epsilon)}{|\mu(a+\epsilon)|} + \sum_j \left[\int_a^{\sigma(a+3\epsilon)} \beta_j(a+\epsilon)\Delta(a+\epsilon)\right]^{\frac{1}{2}} \left[\int_a^{a+3\epsilon} \gamma_j^+(a+\epsilon)\Delta(a+\epsilon)\right]^{\frac{1}{2}}$$

Then for $\Delta(a+\epsilon)=0$, $\beta_j(a+\epsilon)\geq 0$or$\gamma_j(a+\epsilon)\geq 0$ or vice versa.

**Remark4.21.** Note that the second-order equation

$$\sum_j \left(p_j(a+\epsilon)x^{\Delta}\right)^{\Delta}+\sum_j q_j(a+\epsilon)x^{\sigma}=0 \qquad (4.58)$$

can be expressed as an equivalent Hamiltonian system of type (1.1) with

$$\sum_j \alpha_j(a+\epsilon)\equiv 0, \qquad \sum_j \beta_j(a+\epsilon)=\sum_j \frac{1}{p_j(a+\epsilon)}, \qquad \sum_j \gamma_j(a+\epsilon)=\sum_j q_j(a+\epsilon).$$

Therefore, one can easily rewrite the corresponding theorems for (4.58).

**Remark 4.22.** In the special case $\mathbb{T}=\mathbb{Z}$, our results coincide with the corresponding ones in [11], where additionally the stability criteria are also given in connection with Lyapunov-type inequalities when the system is periodic. The stability problem for (1.1) on an arbitrary time scale has been studied in [12].

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