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Research Article

STABILIZATION OF A MATHEMATIC MODEL OF PEST POPULATIONS USING THE TECHNIQUE OF OPTIMAL CONTROL

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Abstract

This paper examines the stabilization of nonlinear differential systems which can be used to model pest interaction. We proceed to apply methods from optimal control theory and design the feedback control by using the Riccati equation to stabilize this system of proposed model equations. Our numerical method will be to test two interesting examples which come from entomological applications. The results show that as the pest population density varies monotonically, calculated steady-states are stabilized using appropriate optimal control techniques.

Keywords: Stabilization, Interaction, Optimal control, Numerical method, Riccati equations, Steady-state, Entomological applications.

1. INTRODUCTION

In the works of Barbu *et al.* (2005) and other several papers cited by these authors, the concept and theory of stabilization of unstable steady state for semilinear parabolic equations has been established within the mathematical sciences. Analytically, we can set up an appropriate algebraic Riccati equations from which we can solve and be able to construct a controller which stabilizes our unstable steady state. Naidu D.S.C (2003) and Patten, (1975). We did not pursue this line of analysis in this paper as we prefer our numerical simulation techniques of checking for the stabilization of unstable steady-state. Our method can be replicated for any other model parameters and subsequently minimize the possibility of incurring approximation errors due to lengthy algebraic calculations. The importance of this technique has been demonstrated in other fields of study (Naidu, 2003), Basak, 2001). In this paper, we want to find out if the unstable steady-state for the system of model equations of ecological competition between two plant species (Ekaka-a, 2009, Ford *et al.,* 2010) can be stabilized by constructing an appropriate controller (Barbu *et al.,* 2005). In our previous and recent experimental analysis Ekaka-a,2009, Ford *et al.,* 2010; Barbi *et al.,* 2005) we know that ecological systems behave like other real world system which are expected to run over a longer period of time to enable clearer qualitative characteristics to be observed. Therefore, an interesting problem in this context is that of stability of ecological systems. In particular, knowledge about the steady-state solutions and stability may provide vital information for ecological studies in predicting the future states of the plant community (Ford *et al.,* 2010). Feedback control laws are also important in ecological studies as they can be used to control the outcome of competition between interacting populations described by a system of coupled nonlinear ordinary differential equations. We propose to use simple linear and nonlinear feedback control schemes to demonstrate some interesting ecological characteristics of the two interacting plant species. In contrast, without control, one of the plant species is more likely to be driven to extinction.

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Since experimental errors are usually characteristic of biological experiments, feedback controller laws can provide useful insights as the lack of precise measurement, of the state variables can be compensated for by including certain parameters in our feedback controller which would provide useful ecological insights. Unstable dynamic system can present interesting problems to scientists and mathematicians (Barbu *et al.,* 2005). In this paper, we would adopt the theoretical definition of stabilization as used in the work of these authors, that is, a controller can be constructed and used to stabilize a given unstable system. This technique has important applications in ecological studies in terms of ecosystem monitoring and decision making (Arnoldand Laub, 1984; Patten, 1975). It is also interesting to acknowledge the fact that ecosystems are rich in feedback and since feedback is the basis of control, we can conceptualize ecosystems as natural controls system (Arnold, 1984). Several references relating to biological control were cited by these authors for interested readers on a further application of control theory. Within the ecological literature, ecosystem stability is an important feature of ecosystems which has made key contributions (Hannon, 1986; Seppelt *et al.,* 1999; Thau, 1972). According to these authors, an ecosystem is said to be stable *if all* independent variables within the dynamical system return to the starting steady-state after a small perturbation from their steady-state. How fast these variables would return to their steady- state is called the concept of resilience. On the contrary, if a dynamical system is unable to return to its steady-state, it is unstable and would have no resilience. In this paper, it is the ecosystem with no resilience that we are attempting to stabilize in order to reach a resilient ecosystem which has relevant ecological applications. What are the methods and results which can be found in the literature about the stabilization and control of interacting populations? In the literature (Barbu *et al.,* 2005) have defined and proved important results about stabilizing some typical semilinear parabolic equations. By using the finite element scheme, these authors have derived control laws to stabilize some examples of semilinear parabolic equations. Other researchers have used the concept of control theory to tackle the control of interacting populations. For example (Vincent, 1972), has used an integral quadratic cost functional to obtain a quasi-optimum feedback control law for two competing species whose dynamics are described by the well-established mathematical formulation of Volterra's competition equations. Next (Vincent *et al.,* 1973) also applied optimal control theory which has an integral linear cost functional to control a prey predictor system described by the Lotka Voterra model equations. Similarly Goh *et al.* (1972) and Yan *et al.* (2009) have studied optimal control or prey-predator systems which are described by the Lotka Volterra equations. More recently Rafikov (2008) have applied the methods from optimal control theory and from the theory of dynamical systems to the mathematical modelling of biological pest control. What are the applications of our present study in ecological studies? The key contribution of our present work is to numerically estimate the depletion rates of two plant species by stabilizing unstable interacting ecological populations. Comparisons of these depletion rates can provide useful information to guide against severe depletion rate which other studies are yet to estimate as far as we know. These results would be of immense application in ecosystem monitoring and decision making against species extinction which would enhance the ideas and norms of ecological services in the sustainability of human life. Since our model equations of competition interaction are unstable, if they are to accurately model real ecosystems, it is inevitable to find the mechanisms of stabilization Hannon (1986). Recently, Ford *et al.* (2010) introduced a mathematical model of plant species interaction in a harsh climate, they considered whether interactions between the species change in character as environment change. The model is constructed based on the notion of a summer season when the plants grow, followed by a winter season when there is no growth but when the plants are subject to the effects of events such as winter storms, see also Ekaka,-a (2009).

The model of competition has the following form:

$$\frac{dy}{dt} = a_1 y(t)(1-b_1 y(t)-\gamma_1 z(t) \qquad (1.1)$$

$$\frac{dz}{dt} = a_2 z(t)(1-b_2 z(t)-\gamma_2 y(t) \qquad (1.2)$$

Here γ and z denote the population of two plant species at time t. Here the non-negative constants a_i $b_{i,}\gamma_i$, i = 1, 2, arc given respectively, as the intrinsic growth rate, the intra- species competitive parameter and the inter- species competitive parameter. These model equations have four steady-state.

$$y = 0, z = 0$$

$$y = 0, z = \frac{1}{b_2}$$

$$y = \frac{1}{b_1}\ z = 0$$

$$y = \frac{b_2 - y_1}{b_1 b_2 - y_1 y_2},\quad z = \frac{b_1 - y_2}{b_1 b_2 - y_1 y_2}$$

The issues on how to choose the parameter values a_i, b_i, γ_i, i =1,2 such that the model is reasonable were also discussed, see also Ford et al (2010). They noticed that although the variation in a_i, b_i, γ_i, i = 1,2 between the species is quite small, the behavior of two such close species are much different over a growing season of several years length. The population of one species may die away and would become extinct over a growing season of several years length. They pointed that small perturbation in the environment could have quite devastating and unexpected results for ecosystems. Some steady-states are stable, but some are not. It is very interesting to design the controller such that the unstable steady-state can be stabilized. There has been significant contribution recently to consider how to stabilize a nonlinear system numerically by using Riccati equation (Yan and Ekaka-a, 2010). Also Barbi (2005) considered how to stabilize a semilinear prarabolic equation numerically. This work has been extendedto the stabilization of the semilinear system and Navier-Stokes equation (Yan *et al.,* 2008; Yan and Tang, 2009). The purpose of this paper is to consider the stabilization of the nonlinear system (1.1) – (1.2) by using Riccati equation, numerically. Our present method is structurally differentfrom the method of (Yan and Ekaka-a (2010).We first linearize the nonlinear system at the unstable steady-state. Then we design the feedback controller for the linearized system by using th Riccati equation. Then we apply the feedback controller to the original nonlinear system. We use the backward Euler method to solve the feedback control system and give the error estimate. We apply our numerical scheme to two different models. For each model, we consider how to stabilize the unstable steady states numerically. The paper is organized as follows. In Section 2, we consider the technique of stabilization of steady-states for a nonlinear system. In Section 3, we consider the method of solution for Stabilizing equations 1.1 and 1.2. In Section 4, we consider two examples to illustrate the technique of stabilization of unstable steady-stales and give some results whereas our results are briefly discussed in section 5.

2. MATHEMATICAL FORMULATION: STABILIZATION OE STEADY-STATES FOR A NONLINEAR SYSTEM

Let us consider the steady-states of the following system of nonlinear first order differential equations

$$\frac{dy}{dt} = y(t)(a_1 - b_1\ y(t) - c_1 z(t) \qquad (2.1)$$

$$\frac{dz}{dt} = z(t)(a_2 - b_1 z(t) - c_2 y(t) \qquad (2.2)$$

with initial conditions y(0) = y_o>0, z(0) = z_0> 0. Here a_i, b_i, c_i,i = 1, 2 are positive constants. The steady-states (y_e, z_e) satisfy

$$y_e\,(a_1 - b_1\ y_e - c_1 z_e) = 0 \qquad (2.3)$$

$$z_e(a_2 - b_2 z_e - c_2\, y_e = 0 \qquad (2.4)$$

which implies that there are four steady-states

$$(y_{e,}\ z_e) = (0,0),\ (y_{e,}\ z_e) = (0, \frac{a_2}{c_2})$$

$$(y_e, z_e) = (\frac{a_1}{b_1}, 0) \text{ and } (y_{e,}\ z_e) = (\frac{a_1 c_2 - c_1 a_2}{b_1 c_2 - c_1 b_2}, \frac{b_1 a_2 - a_1 b_2}{b_1 c_2 - c_1 b_2})$$

To determine the stability of the steady-state (y_e, z_e), we need to consider the. linearized system (2.1) - (2.2) about (y_e z_e). Denote

$$\frac{dy}{dt} = F\ (y,z)$$

$$\frac{dz}{dt} = G\ (y,z)$$

By using Taylor series, we have

$F(y,z) = F(y_e + z_e) + \frac{\partial F(y,z)}{\partial y},(y-y_e) + \frac{\partial F(y,z)}{\partial z}(z - z_e)$ + higher order terms

$G(y,z) = G(y_e + z_e) + \frac{\partial G(y,z)}{\partial y}(y-y_e) + \frac{\partial G(y,z)}{\partial z}(z - z_e)$ + higher order terms

Hence we got the linearized system of (2.1) - (2.2)

$$\frac{dy}{dt} = \frac{\partial F(y,z)}{\partial y}(y-y_e) + \frac{\partial F(y,z)}{\partial z}(z - z_e) \tag{2.5}$$

$$\frac{dz}{dt} = \frac{\partial G(y,z)}{\partial y}(y-y_e) + \frac{\partial G(y,z)}{\partial z}(z - z_e) \tag{2.6}$$

Similarly, $y - y_e$ and $z - z_e$ by Y and Z separately and denoting

$$u = \begin{bmatrix} y \\ z \end{bmatrix} A = \begin{bmatrix} \frac{\partial G(y,z)}{\partial y} & \frac{\partial F(y,z)}{\partial z} \\ \frac{\partial G(y,z)}{\partial y} & \frac{\partial G(y,z)}{\partial z} \end{bmatrix}$$

We have

$$\frac{du}{dt} = Au,\ u(0) = u_{u,} \tag{2.7}$$

where $u_0 = \begin{bmatrix} y_0 & y_e \\ z_0 & z_e \end{bmatrix}$

Lemme 2.1 (Yan and Ekaka-a, 2011): Assume that all the eigenvalues of A arc negative, then the solution of (2.7) tends to the steady state (y_e, z_e) as $t \to \infty$for some suitable initial value $u_0 = (y_0 - y_e, z_o - z_e)$.

This lemma is only stated here without proof. For the proof see the work of Barbu et al (2005) and their related papers.

Remark 2.1. If A has a positive eigenvalue, then the steady state (y_e, z_e) is notstable, i.e., (y(t), z(t)) will not tend to (y_c, z_e) as $t \to \infty$Then we will use the feedback control to stabilize the steady-state.

3. METHOD OF SOLUTION

Equations (1.1) and (1.2) show that the dependent variables arc highly nonlinear of which their solutions are not easily tractable. In the context of this paper, a global approach to such highly nonlinear and coupled ordinary differential equations is the stabilization method. A few of the standard results of stabilizing an unstable steady-state are stated next for the purpose of clarification and their application in our subsequent mathematical analysis.

In this paper, we will only state the following important theorems without proof, for the proofs see the approach in Yan et al (2005), Yan and Ekaka-a (2011)

Theorem 3.1 (Yan and Ekaka-a, (2011): Assume that (y_e, z_e) is unstable, then there exists

$V: (0,\infty) \to R^2$

such that

$$\frac{du}{dt} = Au + Bu,\ u(0) = u_0 \tag{3.1}$$

is exponentially stable at (0, 0).

Here

$V = - R^{-1}B^{*}\prod u$

and $\prod$ satisfies the Riccati equation

$A^{*}\prod + \prod A - \prod BB^{*} + Q = 0$

Here R = I and Q isany positive definite matrix and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

More precisely, there is p > 0, such that for all u_0: $\|u_0\| < \rho$, there exists a unique solution

$u \in C^1(0, +\infty, R^2)$ such that, with some $\gamma > 0$
$\|u(t)\| \leq Ce^{-u}\|u_0\|$

Theorem 3.2 (Yan and Ekaka-a, 2011): Assume that $\begin{bmatrix} y_e \\ z_e \end{bmatrix}$ is unstable then

$$V = - R^{-1} B^{*}\prod \begin{bmatrix} y - y_e \\ z - z_e \end{bmatrix}$$

will stabilize exponentially the nonlinear system

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} F(y,z) \\ G(y,z) \end{bmatrix} + BV(t)$$

More precisely, there exist $\rho > 0$ such that for all $\begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$: $\left\|\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} - \begin{bmatrix} y_e \\ z_e \end{bmatrix}\right\| < \rho$, there exist a unique solution $\begin{bmatrix} y \\ z \end{bmatrix} \in C^1(0, \infty, \Re^2)$, such that, with some constant C and $\gamma > 0$

$$\left\|\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} - \begin{bmatrix} y_e \\ z_e \end{bmatrix}\right\| < Ce^{-u}\left\|\begin{bmatrix} y \\ z_e \end{bmatrix}\right\|$$

4. SOME EXAMPLES

In this section, we shall consider the following examples

4.1. Example 1: The first example is a system of nonlinear first order ordinary differential equations (see Rafikov 2008)

$$\frac{dx_1}{dt} = x_1(t)(y_1 - a_{11})x_1(t) - a_{12}x_2(t) \tag{4.1}$$

$$\frac{dx_2}{dt} = x_2(t)\,(-y_2\, a_{21}x(t) \tag{4.2}$$

Here x_1 and $x_2$represent the population densities of the prey and predator, where $x_1 = 15$ is the initial density of caterpillar population and $x_2 = 2$ is the initial density of parasitoid population. These starting values are chosen for the purpose of our numerical stabilization of optimal control analysis. The positive parameters for this system of equations are

$\gamma_1 = 0.17$, $\gamma_2 = 0.119$, $a_{11} = 0.0003825$, $a_{12} = 0.000935$, $a_{21} = 0.000935$.

What do we want to find out? In this example, we are interested to analyze the following pest control problem

$$\frac{dx_1}{dt} = x_1(t)(\gamma_1 - a_{11}\, x_1(t) - a_{12}x_2(t) \tag{4.3}$$

$$\frac{dx_2}{dt} = x_2(t)(-\gamma_2 + a_{21}x_1(t) + u \tag{4.4}$$

Following Rafikov (2008) the aim of the pest control strategy is to maintain the pest population at level $x_{le} = x_d$ by using a control u_c. In our paper, we have adapted the parameter x_d which is called a pest population density and it is assumed to be below the economic injury level.

To characterize the steady-state solutions of this one prey one predator Lotka- Volterra model, we have solved the above system given $x_{1e} = x_d$ to obtain $x_{2e} = \frac{\gamma_1 - a_{11}x_{1e}}{a_{12}}$ and $u_e = x_{2e}(\gamma_2 - a_{21}x_{1e})$.

This unique system of entomological interaction is fully stabilizable because the rank of the observation matrix is 2 using the concept of optimal control theory (Naidu 2003).
Our first experimental analysis is based on the following parameters:

x_{le} = 20 pests/m^2, x_{2c} = 173.6364, u_e = 17.4157, T *final* = 800 days, x_{10} = 15

x_{20} = 2, the step size k = 0.1. After several variations of final times and initial starting values which did not provide meaningful convergence to steady-state results, we can stabilize the above pest control problem for this set of parameters. Our first main result is graphically presented below.

Next, we shall consider the situation when the pest density threshold level is 10 pests/m^2. Here, our calculated steady-state is (10, 177.7273, 19.4878). By choosing the following parameters X_e = 10 pests/m^2, x_{2e}=177.7273, u_c = 19.4878, T*final* = 1200 days, x_{10} = 15, X_{20} = 2, the step size k = 0.1. By using our technique, we observe that our steady-state can be stabilized. Our second result is presented in the following graph

4.2. Example 2: In this example, we will consider an interesting example of pest-natural enemies interaction model (Guo and Chen, 2009). The uncontrolled model without impulsive harvest parameters was formulate by the following system of first order coupled differential equations.

$$\frac{dx}{dt} = x(t)(a - by(t)) \quad (4.5)$$

$$\frac{dy}{dt} = y(t)(cx(t) - d) \quad (4.6)$$

where x(t) represents the number of pests at time t and y(t) represents the number of natural enemies at time t. It is worth mentioning that natural enemies make an important contribution in limiting potential pest populations (Rafikov 2008).

FIGURE 2. Uncontrolled and Controlled Solution Trajectories of Steady-Stale (10.0001, 177.7273)

Table 1. Calculation of convergence of steady-states over a time interval 4.2

Examples									
				Other Results					
No	k	x_d	x_{1c}	x_{2c}	u_e	T *final*	x_{10}	x_{20}	Point
3	0.1	18	18	174.45	17.82	500	15	2	(18,174.45)
4	0.1	16	16	175.27	18.23	800	15	2	(16, 175.27)
5	0.1	14	14	176.09	18.65	900	15	2	(14, 176.09)
6	0.1	12	16	176.90	19.06	1000	15	2	(16, 176.90)
7	0.1	15	15	175.68	18.44	1000	15	2	(15, 175.68)
8	0.1	25	25	171.59	16.408	900	15	2	(25, 171.59)
9	0.1	35	35	167.5	14.45	700	15	2	(35, 167,5)
10	0.1	45	45	163.409	12.57	600	15	2	(45, 163.40S

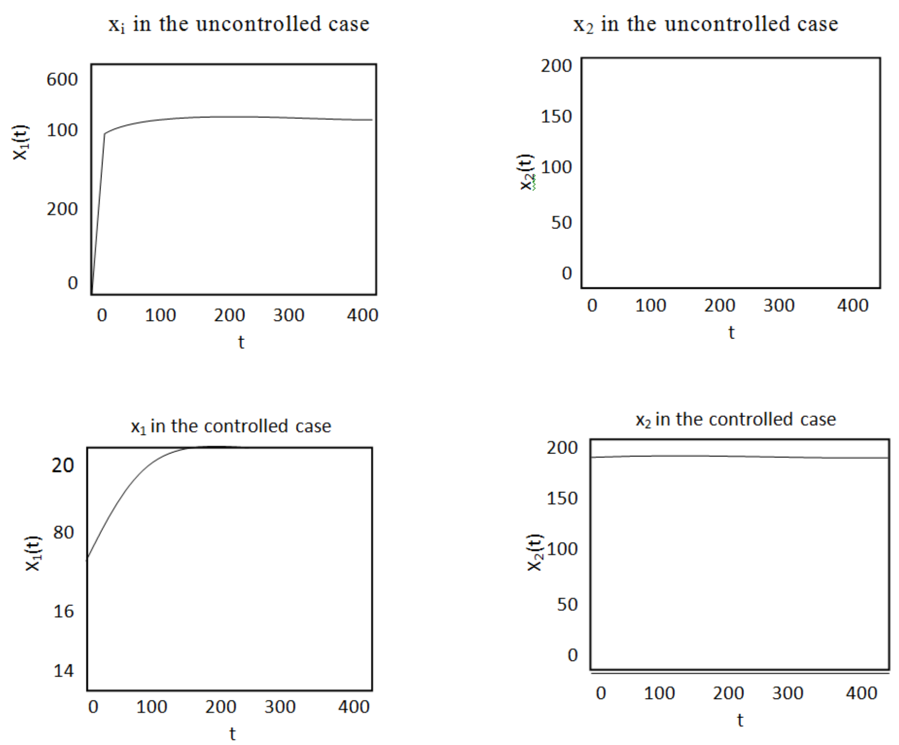

Figure 1. Uncontrolled and Controlled Solution Trajectories of Steady-State (20.0000, 173.6364)

Table 2. Calculation of convergence of steady-states over a time interval

Examples		Other Results							
No	k	PP[1] .	Xlc	X2c	Uc	T *final*	x |()	X20	Point
Z.	0.1	0.8	0.8	2.5	0.3	100	15	2	(0.8, 2.5)
3	0.1	0.85	0.85	2.5	0.2875	100	15	2	(0.85, 2.5)
4	0.1	0.9	0.9	2.5	0.275	100	15	2	(0.9, 2.5)
5	0.1	0.95	0.95	2.5	0.2625	100	15	2	(0.95, 2.5)
6	0.1	0.5	0.5	3.3333	0.5	100	15	2	(0.5, 3.3333)
7 -	0.1	0.8	0.8	3.3333	0.4	100	15	2	(0.8, 3.3333)
8	0.1	0.85	0.85	3.3333	0.3833	100	15	2	(0.85,3.3333)
9	0.1	0.85	0.85	5	1.5757	100	15	2	(0.85,5)
10	0.1	1.5	1.5	5	1.25	100	15	2	(1.5, 5)

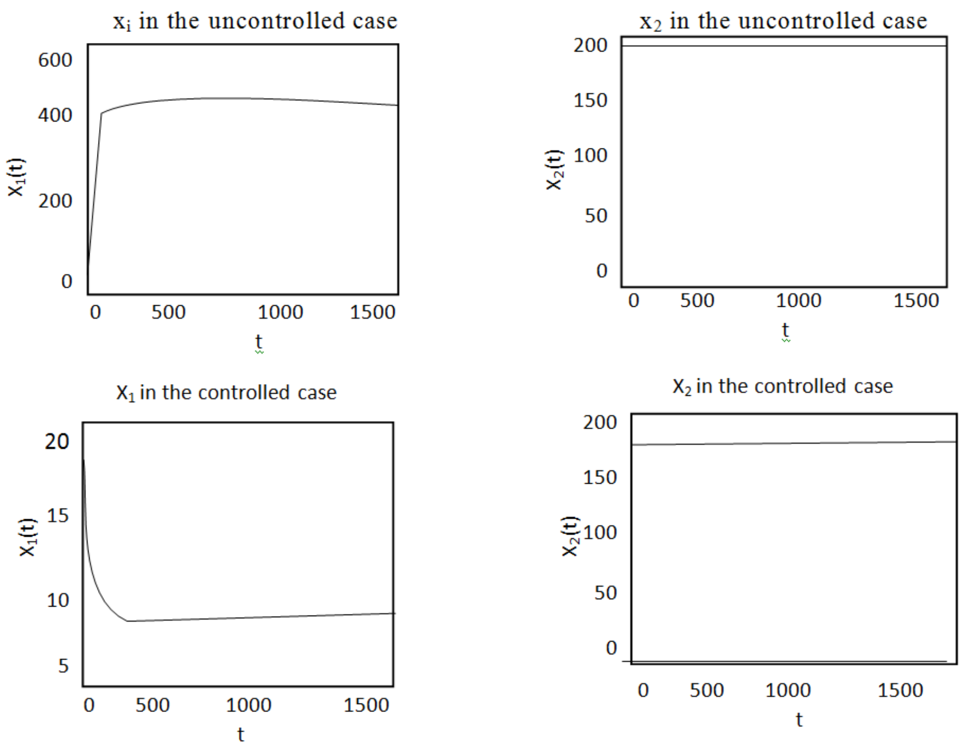

Figure 2. Uncontrolled and Controlled Solution Trajectories of Steady-Stale (10.0001, 177.7273)

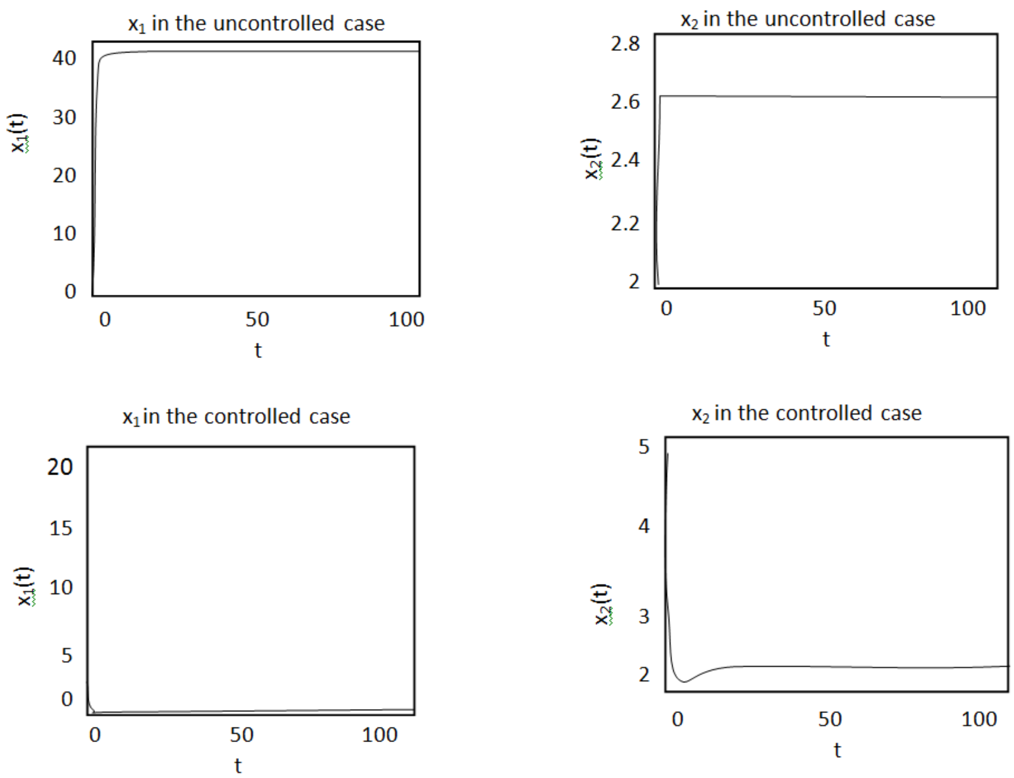

Figure 3.

Our aim is to attempt to construct a stabilizer which can stabilize the corresponding control problem using our optimal control numerical technique. Here, we will like to analyze the following Lotka-Volterra initial control problem

$$\frac{dx}{dt} = x(t)(a - by)(t) \quad (4.7)$$

$$\frac{dy}{dt} = y(t)(cx(t) - d) + u \quad (4.8)$$

Under the following simplifying assumptions

- The prey in the absence of any predation will tend to grow unboundedly in a Malthusian pattern; this pattern of growth is reflected in the ax term (Guo and Chen, 2009)
- The impact of predation would be to reduce the prey's per-capita growth rate by a term which is proportional to both the prey and the predator populations' this pattern of growth is reflected in the bxy term (Guo and Chen, 2009)
- In the absence of any prey, the predators death rate would tend to decay exponentially which is reflected in the dy term (Guo and Chen, 2009),
- The prey's effect to the predator's growth rate is cxy; what this pattern of growth means is that the preys contribution would be proportional to the available prey as well as to the size of the predator population (Guo and Chen, 2009).

By solving the above system of equations in Example 2 at steady-state condition, we will obtain the steady-state (x_e, y_e) where xe = per population, $y_e = \frac{a}{b}$

Here, uc - $y_e(d - cx_c)$–this this study, the notation pre population is called the pest population level (Rafikov, 2009).

Next, we are interested to investigate the extent of stabilizing the control problem using the same optimal control technique applied in Example 1. The next key result is presented below for both the uncontrolled and controlled scenarios. Here we have used the following parameters to check for the convergence of the steady-state: a =5, b = 2, c = 0.1, d = 0.2, x_c = 0.5, y_e = 2.5, u_c = 0.3754, x(0) = 15 y(0) = 2, T*final* = 1000 These further variations of Example 2 show the convergence to calculated steady state which reinforce similar ecological insights which we have mentioned in Example 1.

5. DISCUSSION OF RESULTS

In these two experimental examples, we have found that as the pest population density varies monotonically, calculated steady-states can be stabilized using our optimal control technique. In these scenarios, we have been able to maintain the pest population at a level $X_i = x_d$ by using a control u_e where x_d is a pest population density below the economic injury level. Our set of results further authenticate one of the dominant short-term pest control strategies which are cost effective and can provide further insights into efficient agro-ecology functioning as a result of preserving biodiversity. What is the significance of this analysis in entomogical study? When the pest population is below the economic injury level, it means that the agro-ecology is functioning efficiently as a result of preserving biodiversity. In this situation natural enemies of insect pests such as predators, parasitoids, and pathogens (Rafikov 2008) are able to keep the population of pest in check and there will be no need to apply pesticides. Hence, our method of controlling the pest population by designing a controller with which to stabilize a biological pest population is attractive and cost effective. Despite these key contribution pest population densities are never always constant due to disturbances in the ecological equilibrium. When this happens, some control must be initiated to return the expected ecological balance to equilibrium and thus enable natural enemies to continue their work.

6. CONCLUDING REMARKS AND FURTHER RESEARCH

We have systematically achieved in this study that for an arbitrary pest population density which is taken to be below the economic injury level, the chosen steady-state solution for the posed problem can be stabilized. We would expect these series of results to provide insights for pest control intervention strategy subject to a few error estimates which were not defined and analysed in this present paper.

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